Riemann hypothesis and quantum mechanics.

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- The **Riemann hypothesis** implies results about the **distribution of prime numbers**. Along with suitable generalizations, some mathematicians consider it the most important unresolved problem in pure mathematics (Bombieri 2000 [1]). The Riemann hypothesis, along with Goldbach's conjecture, is part of Hilbert's eighth problem in David **Hilbert's list of 23 unsolved problems**; it is also one of the **Clay Mathematics Institute's Millennium Prize Problems**.
- 1 Solved problems

  1.1 Poincaré conjecture

  2 Unsolved problems

  2.1 P versus NP **2.2** Hodge conjecture **2.3** Riemann hypothesis **2.4** YangMills existence and mass gap **2.5** NavierStokes existence and smoothness **2.6** Birch and Swinnerton-Dyer conjecture

[1] Bombieri, Enrico (2000), The Riemann Hypothesis official problem description (PDF), Clay Mathematics Institute, retrieved 2008-10-25 Reprinted in (Borwein et al. 2008).

Riemann zeta function and Riemann hypothesis

- The Riemann zeta function

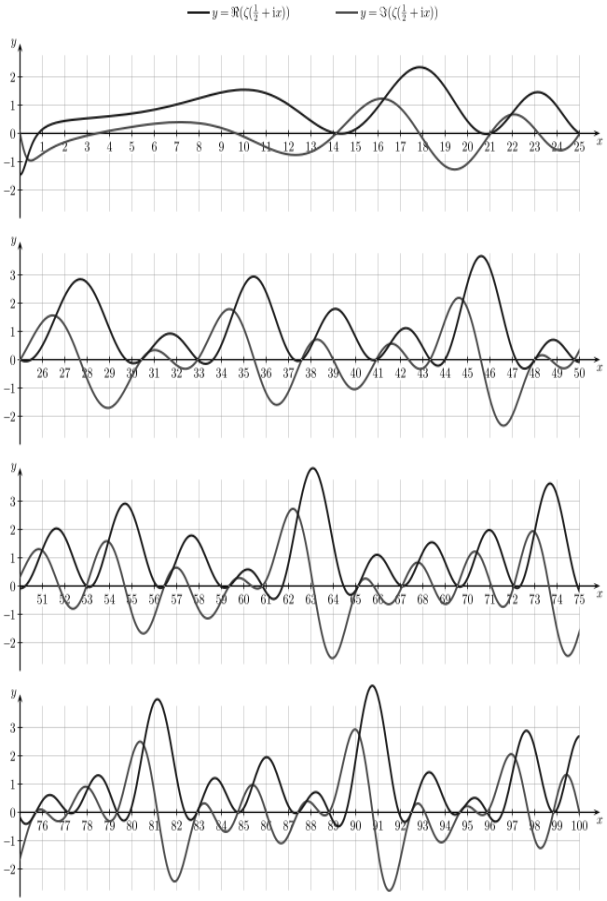
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}, \quad \Re(s) > 1$$

[2] may be extended to the whole $\mathbb{C}$ through
$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\xi(s) = \xi(s-1)$.

- The function $\zeta(s)$ has (trivial) zeros at $-2, \ -4, \cdots$. The other zeros are of the form $\frac{1}{2} + i\alpha$ when α is a zero of $\xi(s)$. Riemann hypothesis, 1859 (RH) is the statement that **the nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.**

[2] $\Gamma(s) = \int_1^{\infty} x^{s-1} e^{-x} dx$, $\Gamma(n) = (n-1)!$

First critical Riemann zeros



yearn	author
1903	15 J. P. Gram
1914	79 R. J. Backlund
1925	138 J. I. Hutchinson
1935	1041 E. C. Titchmarsh
1953	1104 A. M. Turing
1956	15000 D. H. Lehmer
1956	25000 D. H. Lehmer
1958	35337 N. A. Meller
1966	250000 R. S. Lehman
1968	3500000 J. B. Rosser, J. M. Yohe, L. Schoenfeld
1977	40000000 R. P. Brent
1979	81000001 R. P. Brent
1982	200000001 R. P. Brent, J. van de Lune, H. J. J. te Riele, D. T. Winter
1983	300000001 J. van de Lune, H. J. J. te Riele
1986	1500000001 J. van de Lune, H. J. J. te Riele, D. T. Winter
2001	10000000000 J. van de Lune (unpublished)
2004	900000000000 S. Wedeniwski
2004	10000000000000 X. Gourdon and P. Demichel

Introduction [3]

- Let $\Psi(n) := n\prod_{p|n}(1+\frac{1}{p})$ denote the **Dedekind Ψ function**. Define, for $n \geq 3$, the ratio $R(n) := \frac{\Psi(n)}{n \log\log n}$. We prove unconditionally that $R(n) < e^{\gamma}$ for $n \geq 31$. Let $N_n = 2 \cdots p_n$ be the primorial of order n. We prove that the statement $R(N_n) > \frac{e^{\gamma}}{\zeta(2)}$ for $n \geq 3$ is equivalent to the Riemann Hypothesis.

[3] P. Solé and M. Planat, Extreme values of the Dedekind psi function,J. Combinatorics and Number Theory, **1** (3), 1-6 (2011).

Dedekind 1

- The **Dedekind** $\psi(n)$ is related to the **sum of divisor function**

$$\sigma(n) = \sum_{d|n} d$$

by the inequalities

$$\Psi(n) \leq \sigma(n),$$

and the fact that they coincide for n squarefree.

- It is also related to **Euler** φ **function** by the inequalities

$$n^2 > \varphi(n)\Psi(n) > \frac{n^2}{\zeta(2)}$$

derived as Proposition/Theorem 5 of the paper.

Euler's constant

- Euler's constant is defined as $\gamma = \lim_{n\to\infty}(-\ln(n) + \sum_{k=1}^{\infty} \frac{1}{k})$.
- $\gamma = 0.577215664901532\cdots$
- $\zeta(2) = \frac{\pi^2}{6}$.

Dedekind 2

- In view of the studies of **large values of** σ [4] and **of low values of** φ [5], it is natural to study both the large and low values of Ψ. We define the ratio $R(n) := \frac{\Psi(n)}{n \log\log n}$ (Proposition 3 below).
- We prove unconditionally that $R(n) < e^{\gamma}$, for $n \geq 31$. This bound would follow also **from Robin inequality** $\sigma(n) \leq e^{\gamma} n \log\log n$ for $n \geq 5041$ under Riemann Hypothesis (RH), since $\Psi(n) \leq \sigma(n)$.
- In the direction of lower bounds, we prove that the statement $R(N_n) > \frac{e^{\gamma}}{\zeta(2)}$ for $n \geq 3$ is equivalent to RH, where $N_n = 2 \cdots p_n$ is the primorial of order n. The proof relies on **Nicolas's work on the Euler totient function**.

[4] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. J. Math. Pures Appl. (9) 63 (1984), 187–213.

[5] Nicolas, Jean-Louis Petites valeurs de la fonction d'Euler. J. Number Theory 17 (1983), no. 3, 375–388.

Dedekind 3: $\psi(n)$ at primorial numbers

- **Proposition 3**: We have, as $n \to \infty$ $\lim R(N_n) = \frac{e^\gamma}{\zeta(2)} \approx 1.08$.

  Proof; Writing $1 + 1/p = (1 - 1/p^2)/(1 - 1/p)$ in the definition of $\Psi(n)$ we can combine the Eulerian product for $\zeta(2)$ with Mertens formula $\prod_{p \leq x}(1 - 1/p)^{-1} \sim e^\gamma \log(x)$ to obtain

$$\frac{\Psi(N_n)}{N_n} \sim \frac{e^\gamma}{\zeta(2)} \log(p_n),$$

  Now the Prime Number Theorem [6] states that $x \sim \theta(x)$ for x large. where $\theta(x)$ stands for Chebyshev's first summatory function:

$$\theta(x) = \sum_{p \leq x} \log p.$$

  This shows that, taking $x = p_n$ we have $p_n \sim \theta(p_n) = \log(N_n)$. The result follows.

[6] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers.*, Oxford, 1979

Dedekind 4: lower bounds

We reduce first to Euler's φ function.

- **Proposition 5**: For $n \geq 2$ we have $n^2 > \varphi(n)\Psi(n) > \frac{n^2}{\zeta(2)}$.
- **Theorem 4.1**: Under RH the ratio $R(N_n)$ is $> \frac{e^\gamma}{\zeta(2)}$ for $n \geq 3$. If RH is false, this is still true for infinitely many n.
- Conjecture 4.1: For all $n \geq 3$ we have $R(N_n) > \frac{e^\gamma}{\zeta(2)}$.
- Theorem 4.2: Conjecture 4.1 is equivalent to RH.

Dedekind 5: Proof of theorem 4.2

We reduce first to Euler's φ function.

- Theorem 4.2: Conjecture 4.1 is equivalent to RH.
- **If RH is true** we refer to the first statement of Theorem 4.1. **If RH is false** we consider the function

$$g(x) := \frac{e^\gamma}{\zeta(2)} \log \theta(x) \prod_{p \le x} (1 + 1/p)^{-1},$$

Observing that $\log \theta(p_n) = \log \log N_n$, we see that $g(p_n) < 1$ is equivalent to $R(N_n) > \frac{e^\gamma}{\zeta(2)}$. We need to show that there exists an $x_0 \ge 3$ such that $g(x_0) > 1$ or equivalently $\log g(x_0) > 0$. Using once again the identity $1 - 1/p^2 = (1 - 1/p)(1 + 1/p)$, and Choi et al (Lemma 6.4), from

$$-\log \zeta(2) = \sum_{p \le x} \log(1 - 1/p^2) + \sum_{p > x} \log(1 - 1/p^2),$$

one gets the bound $\log g(x) \ge \log f(x) - 2/x$, where f is the function $f(x) := e^\gamma \log \theta(x) \prod_{p \le x} (1 - 1/p)$. introduced in Nicolas (1983).

- We know by Nicolas (Theorem 3) that, if RH is false, there is a $0 < b < 1$ such that $\limsup x^{-b} f(x) > 0$ and hence $\limsup \log f(x) >> \log x$. Since $2/x = o(\log x)$, the result follows.

Introduction [8]

- The **Pauli group** $\mathcal{P}_q$ is generated by quantum generators X (shift) and Z (clock) in **the q-dimensional Hilbert space**. The number of maximal mutually commuting sets within $\mathcal{P}_q$ is controlled by the Dedekind psi function $\psi(q)$. There exists a specific inequality involving the Euler constant $\gamma \sim 0.577$, that is only satisfied at specific low dimensions $q \in \mathcal{A} = \{2, 3, 4, 5, 6, 8, 10, 12, 18, 30\}$. The set $\mathcal{A}$ is closely related to the set $\mathcal{A} \cup \{1, 24\}$ of integers that are **totally Goldbach**, i.e. that consist of all primes $p < n - 1$ with p not dividing n and such that $n - p$ is prime [7].
- **In the extreme high dimensional case**, at primorial N_r, one introduces the Hardy-Littlewood function $R(q)$ for estimating the number of Goldbach pairs, and one establishes a new inequality (Theorem 4) for the equivalence to Riemann hypothesis in terms of $R(N_r)$.

[7] Brouwers D G, Bamberg J. and Cairns G., *Aust. Math. Soc. Gaz.* **31** 251-55 (2004).

[8] M. Planat, F. Anselmi and P. Solé, Theoretical and Mathematical Physics, **171** (3) 780-791 (2012).

Totally Goldbach numbers

Goldbach's conjecture, formulated in 1742, is that every even integer greater than 2 is the sum of two primes. To date it has been checked for q up to 2.10^{18} [9].

- **Definition 1**: A positive integer n is **totally Goldbach** if for all primes $p < n - 1$, with p not dividing n (except when $p = n - p$) we have that $n - p$ is prime (Brouwers, 2004). The inequality
$$\frac{\psi(q)}{q} > e^{\gamma} \log \log q$$
is only satisfied at a totally Goldbach number $q \in \mathcal{A} = \{2, 3, 4, 5, 6, 8, 10, 12, 18, 30\}$. The only totally Goldbach numbers not satisfying the inequality are $q = 1$ and $q = 24$.

[9] Oliveira e Silva T 2011 Goldbach conjecture verification. http://www.ieeta.pt/~ tos/goldbach.html

The Dedekind psi number theoretical inequality

- **Theorem 1** Solé and Planat, 2011: Let $N_r = 2 \cdots p_r$ be the primorial of order r. The statement $\frac{\psi(N_r)}{N_r \log\log N_r} > \frac{e^\gamma}{\zeta(2)}$ for every $r > 2$ is equivalent to the Riemann hypothesis.
  Below we establish further statements relating $\psi(q)$ and the Hardy-Littlewood function $R(q)$ about the number of Goldbach pairs. An important result is conjecture 1 that is found to be equivalent to RH by theorem 4.

The maximum commuting sets of qudits

- A qudit is defined as the group $\mathcal{P}_q = \langle X, Z \rangle$ with unitary quantum generators and $\omega = \exp(\frac{2i\pi}{q})$

$$X = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad Z = \mathrm{diag}(1, \omega, \omega^2, \dots, \omega^{q-1}),$$

- **Theorem 2** (Planat, 2010): Let $\mathfrak{S}$ be the set of maximal mutually commuting sets $\mathfrak{M}$ in $\tilde{\mathcal{P}}_q$. The cardinality of $\mathfrak{M}$ is $|\mathfrak{M}| = q - 1$ and that of $\mathfrak{S}$ is $|\mathfrak{S}| = \sigma(q)$, where $\sigma(q)$ is the sum of divisor function.
- **Theorem 3** (Planat, 2010): Let $\mathfrak{S}$ be the set of maximal mutually commuting sets $\mathfrak{M}$ in $\tilde{\mathcal{P}}_q$ and $\mathfrak{D} \subset \mathfrak{S}$ the subset of $\mathfrak{S}$ such that for any $d \in \mathfrak{D}$ and $s \in \mathfrak{S}$ the intersection of d and s is non empty, i.e. $d \cap s \neq \emptyset$. The cardinality of $\mathfrak{S} - \mathfrak{D}$ is the Dedekind psi function $\psi(q)$.

On Goldbach pairs at primorial dimensions and RH: 1

The number of Goldbach pairs (p_1, p_2) of primes whose sum is the even integer q is the Hardy-Littlewood function $g(q) = R(q)\frac{q}{\log^2 q}$, with $R(q) = 2C_2 \prod_{p|q} \frac{p-1}{p-2}$, and $C_2 = \prod_{p>2}(1 - \frac{1}{(p-1)^2}) \sim 0.660$ is the twin prime constant (Hardy and Littlewood, 1922).

- **Proposition 1:** Let $N_r = 2 \cdots p_r$ be the primorial number of order r, we have $\lim_{r\to\infty} \frac{R(N_r)}{\log\log N_r} = e^\gamma$.
- **Conjecture 1:** The Hardy-Littlewood inequality $\frac{R(N_r)}{\log\log N_r} > e^\gamma$ holds for every $r > 1$.

r	N_r	$u_r - e^\gamma$
2	6	2.74
10	6.5×10^9	0.23
10^2	4.2×10^{219}	0.028
10^3	6.8×10^{3392}	0.0049
10^4	9.1×10^{45336}	0.0010
10^5	1.9×10^{563920}	0.00023

On Goldbach pairs at primorial dimensions and RH: 2

- **Proposition 2:** For an even integer q, the following inequality holds
$$R(q) < \zeta(2)\frac{\psi(q)}{q}.$$
- **Corollary 1:** The primorial numbers N_r are champion numbers, i.e. left to right maxima of the function $q \to x(q) = \frac{1}{\zeta(2)}\frac{R(q)}{\psi(q)/q}$. One gets $\lim_{r\to\infty} x(N_r) = 1$.
- **Theorem 4:** Conjecture 1 is equivalent to RH.
Proof: If RH is not true, then there are infinitely many r such $\frac{R(N_r)}{\log\log N_r} < e^{\gamma}$, corresponding to as many violations of the Hardy-Littlewood inequality in Conjecture 1.
If RH is true, this follows from the study of function
$$g(x) = \frac{e^{\gamma}}{2C_2}\log\theta(x)\prod_{p\leq x}\frac{p-2}{p-1}$$
One proves that under RH, $g(x) < 1$.

Abstract [11]

- There exists a **a quantum dynamical system whose partition function is the Riemann zeta function** $\zeta(\beta)$ [10]. We formulate Riemann hypothesis (RH) as a property of the low temperature Kubo-Martin-Schwinger (KMS) states of this theory. More precisely, the expectation value of the BC phase operator is $\phi_\beta(q) = N_{q-1}^{\beta-1}\psi_{\beta-1}(N_q)$, where $N_q = \prod_{k=1}^{q} p_k$ is the primorial number of order q and $\psi_b(q)$ a generalized Dedekind ψ function depending on one real parameter b as $\psi_b(q) = q\prod_{p\in\mathcal{P},p|q}\frac{1-1/p^b}{1-1/p}$. Fix $\beta > 2$, **RH is then shown to be equivalent to**

$$N_q|\phi_\beta(N_q)|\zeta(\beta-1) > e^\gamma \log\log N_q,$$

for q large enough. Under RH, extra formulas for high temperatures KMS states ($1.5 < \beta < 2$) are derived.

[10] Bost J C and Connes A 1995 Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory *Sel. Math.* **1** 411-457.

[11] M. Planat, P. Solé and S. Omar, J. Phys. A: Math. Theor. **44**, 145203 (2011).

Introduction 1

- The Riemann Hypothesis (RH), that describes the non trivial zeroes of Riemann ζ function, is a Holy Grail of Mathematics. Many formulations of RH may be found in the literature. In this paper, we mainly refer to the Nicolas inequality [12]

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma} \log\log N_k,$$

where $N_k = \prod_{i=1}^{k} p_i$ is the primorial of order k, φ is the Euler totient function and $\gamma \approx 0.577$ is the Euler-Mascheroni constant.

- The above inequality **holds for all** $k \geq 1$ **if RH is true**. **Conversely, if RH is false**, the inequality holds for infinitely many k, and is **violated** for infinitely many k.

[12] Nicolas J L 1983 Petites valeurs de la fonction d'Euler *J. Number Theory* **17** 375-388.

Introduction 2

- The Bost and Connes (BC) system is a quantum dynamical model whose **partition function is the Riemann zeta function** $\zeta(\beta)$ (with $\beta = \frac{1}{kT}$ the inverse temperature) and that exhibits a **spontaneous symmetry breaking at the pole** $\beta = 1$.
- The behaviour of the system reminds that of a ferromagnet. At a temperature larger than the critical temperature $T_c \approx 10^3 K$, the disorder of the spins dominates and the thermal equilibrium state is unique while for $T < T_C$ the individual spins tend to align to each other, which in the three-dimensional space $\mathbb{R}^3$ of the ferromagnet yields a set of equilibrium thermal states parametrized by the symmetry group $SO(3)$ of rotations in $\mathbb{R}^3$.
- In BC approach, the Hamiltonian operator is the **logarithm of the number operator** N as $H_0 = \log N$. This contrasts with the ordinary Hamiltonian $H_0 = N + \frac{1}{2}$ for the harmonic oscillator.

Introduction 3

- First, we rewrite the abstract mathematical language of BC theory with, as much as possible, concepts of **quantum information processing**.
- Second, we uncover a relationship between the statistical phase states (KMS states) of BC theory and Riemann hypothesis (RH). The Riemann hypothesis is shown to be **equivalent to an inequality** satisfied by the KMS states of the BC theory at a selected low temperature $\beta > 2$. **A special case** (at $\beta = 3$) of this inequality is $\psi(N_k)/(N_k \log\log N_k) > e^{\gamma}/\zeta(2)$ for $k > 3$, where $\psi(n) = \psi_2(n)$ is **the Dedekind ψ function** (Solé & Planat 2010).
- Higher temperatures KMS states ($1.5 < \beta < 2$) require RH to be analyzed with conventional tools such as the prime counting function $\pi(x)$. Very high temperatures KMS states ($0 < \beta < 1.5$) are left as an open problem.

On modular exponentiation: classical

- There is a **the parallel between the BC theory and QIP** with the problem of finding the multiplicative order r in the modular equation

$$a^x = 1 \pmod{q}. \tag{1}$$

Given the integer a and a positive integer q with $(a, q) = 1$, the order $r = \mathrm{ord}_q(a)$ is the smallest positive integer x satisfying (1). It is straigthforward to establish the set of inequalities

$$\mathrm{ord}_q(a) \leq \lambda(q) \leq \varphi(q) \leq q - 1, \tag{2}$$

where $\varphi(q)$ is the Euler totient function [it counts the number of irreducible fractions $\frac{p}{q}$ with $p < q$] and $\lambda(q)$ is the Carmichael lambda function [it is the smallest positive integer x satisfying (1) for *every* integer $a < q$ with $(a, q) = 1$] [13].

[13] The recursive calculation of $\lambda(n)$ is as follows. For a positive integer $k \geq 3$, $\lambda(2^k) = 2^{k-2}$. For p an odd prime ≥ 3 or $0 < k \leq 2$, $\lambda(p^k) = \varphi(p^k) = (p-1)p^{k-1}$. For $q = \prod_{i=1}^{n} p_i^{k_i}$ a product of primes, $\lambda(q) = \mathrm{lcm}[\lambda(p_i^{k_i})]_i$, where lcm denotes the least common multiple of the numbers in the bracket.

On modular exponentiation: quantum

- **Order-finding is a task that can in principle be performed more efficiently on a quantum computer than on a classical computer** (in the Schor algorithm).

  Let q be an m-bit integer and the integer $a < q$ with $(a, q) = 1$. The quantum algorithm for order-finding is based on the following unitary operator acting on m-bits and such that for any $n < q$

$$|n\rangle \to U_a |n\rangle = |an \bmod q\rangle . \tag{3}$$

  The eigenvalues of the operator (3) are in the form $\exp(2i\pi\frac{k}{r})$ and the corresponding eigenvectors are given as a QFT

$$|u_k\rangle = \frac{1}{r^{1/2}} \sum_{j=0}^{r-1} \exp(\frac{-2i\pi kj}{r}) \left|a^j \bmod q\right\rangle . \tag{4}$$

  The efficient implementation of the QFT on a quantum computer allows the efficient estimation of the period r of the function $f(x) = a^x \bmod q$ in (1), and subsequently the efficient factorization of an integer (in polynomial time instead of the exponential time required on a classical computer).

On the KMS-states of Bost-Connes model: 1

- BC theory is a skilfull **balance of concepts related to QIP and to quantum statistical physics** . The BC $\mathbb{C}^\star$ algebra is the cross product

$$\mathbb{C}_{\mathbb{Q}} = \mathbb{C}^\star(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\star \tag{5}$$

of the $\mathbb{C}^\star$-algebra $\mathbb{C}^\star(\mathbb{Q}/\mathbb{Z})$ of additive/phase operators e_δ acting on qudits $|n\rangle$ as

$$|n\rangle \to e_\delta |n\rangle = \exp(2i\pi \frac{p}{q}) |n\rangle , \quad \delta = \frac{p}{q} \in \mathbb{Q}/\mathbb{Z}, \tag{6}$$

by the semigroup $\mathbb{N}^\star$ of multiplicative operators μ_a

$$|n\rangle \to \mu_a |n\rangle = |an \bmod q\rangle , \quad a \in \mathbb{N}^\star. \tag{7}$$

The operator μ_a is closely related to U_a but acts on qudits $|n\rangle$.

On the KMS-states of Bost-Connes model: 2

- The BC system is the pair $(\mathbb{C}_{\mathbb{Q}}, \sigma_t)$ with entries the algebra $\mathbb{C}_{\mathbb{Q}}$ and a quantum statistical flow σ_t. [14] For the more general case of an algebra of observables A, the Gibbs state is replaced by the Kubo-Martin-Schwinger (or KMS_β state). One introduces a 1-parameter group σ_t of automorphisms of A so that $\forall t \in \mathbb{R}$ and $x \in A$

$$\sigma_t(x) = \exp(itH_0)x\exp(-itH_0). \tag{8}$$

For the algebra A, the equilibrium state remains unique only if some conditions regarding the evolution of the operator $x \in A$ are satisfied.

[14] In standard quantum statistical mechanics, given an observable Hermitian operator M and a Hamiltonian H_0, one has the evolution $\sigma_t(M)$ versus time t as

$$\sigma_t(M) = \exp(itH_0)M\exp(-itH_0),$$

and the expectation value of M at the inverse temperature $\beta = \frac{1}{kT}$ is the unique Gibbs state

$$\mathrm{Gibbs}(M) = \mathrm{Trace}(M\exp(-\beta H_0))/\mathrm{Trace}(\exp(-\beta H_0).$$

The KMS flow of BC system: 1

- **The Hamiltonian on the BC system** is defined from its action on the qudits $|n\rangle$ by

$$H_0 \, |n\rangle = \ln n \, |n\rangle \, . \tag{9}$$

Using the relations $\exp(-\beta H_0) \, |n\rangle = \exp(-\beta \ln n) \, |n\rangle = n^{-\beta} \, |n\rangle$, it follows that **the partition function** of the model at the inverse temperature β is

$$\mathrm{Trace}(\exp(-\beta H_0)) = \sum_{n=1}^{\infty} n^{-\beta} = \zeta(\beta), \quad \Re(\beta) > 1. \tag{10}$$

Applying (8) to the Hamiltonian (9) one obtains

$$\sigma_t(\mu_a) = a^{it} \mu_a \text{ and } \sigma_t(e_\delta) = e_\delta \text{ for any } a \in \mathbb{N}^\star, \ \delta \in \mathbb{Q}/\mathbb{Z} \text{ and } t \in \mathbb{R}. \tag{11}$$

The KMS flow of BC system: 2

- It can be shown that in the high temperature regime $0 < \beta \leq 1$ there is a unique KMS_β state of the quantum statistical system $(\mathbb{C}_\mathbb{Q}, \sigma_t)$. It corresponds to the partition function (10). In the low temperature regime $\beta > 1$, the KMS_β is no longer unique. The symmetry group G of the quantum statistical system is a subgroup of the automorphisms of $(\mathbb{C}_\mathbb{Q}, \sigma_t)$ which commutes with σ_t, i.e.

$$w \circ \sigma_t = \sigma_t \circ w \text{ for any } w \in G \text{ and } t \in \mathbb{R}, \tag{12}$$

where $\circ$ is the composition law. The symmetry group G permutes a family of extremal KMS_β states generating the possible states of the system after phase transition. **The symmetry group of the algebra (μ_a, e_δ) endowed with the Hamiltonian H_0 in (9) is the Galois group $G = \mathrm{Gal}(\mathbb{Q}^{\mathrm{cycl}}/\mathbb{Q})$ of the cyclotomic extension $\mathbb{Q}^{\mathrm{cycl}}$ of $\mathbb{Q}$.** The field $\mathbb{Q}^{\mathrm{cycl}}$ is generated in $\mathbb{Q}$ by all the roots of unity. The Galois group G is $\mathrm{aut}(\mathbb{Q}^{\mathrm{cycl}})$. It is isomorphic to $\mathbb{Z}_q^\star$.

The KMS flow of BC system: 3

- At any temperature $\beta < \infty$, the restriction of the KMS$_\beta$ state to $\mathbb{C}^\star(\mathbb{Q}/\mathbb{Z})$ is given by

$$\phi_\beta(q) = q^{-\beta} \prod_{p \in \mathcal{P}, p|q} \frac{1 - p^{\beta-1}}{1 - p^{-1}} \tag{13}$$

and for $\beta = \infty$ it is

$$\phi_\infty(q) = \mu(q)/\varphi(q), \tag{14}$$

where μ is the Möbius function.

The KMS state (13) may be easily rewritten as $\phi_\beta(q) = N_{q-1}^{\beta-1}(q)\psi_{\beta-1}(q)$, where $N_q = \prod_{k=1}^{q} p_k$ is a primorial number and the **generalized Dedekind psi function** $\psi_b(q)$ is defined in the abstract.

The Bost-Connes KMS states and Riemann hypothesis

- One may formulate Riemann hypothesis as a property of the KMS states (13) with the inequality

$$\epsilon_\beta(q) = \frac{N_q|\phi_\beta(N_q)|}{\log\log N_q} - \frac{e^\gamma}{\zeta(\beta-1)} > 0 \text{ when } \beta > 2. \qquad (15)$$

The special case $\beta = 3$ corresponds to paper Solé & Planat, 2010. A concise account of numerical values at selected temperature are in table 1.

q	10	10^2	10^3	10^4
N_q	6.4×10^{10}	4.7×10^{219}	6.7×10^{3392}	9.1×10^{45336}
$\epsilon_{2.1}$	0.25	0.093	0.051	0.031
ϵ_3	0.16	0.018	3.0×10^{-3}	6.1×10^{-4}
ϵ_{10}	0.25	0.028	4.9×10^{-3}	1.0×10^{-3}

Table: A check of KMS inequality (15).

The Bost-Connes KMS states and Riemann hypothesis: 2

- **Proposition 1:** Pick a real number $b > 1$. and denote $R_b(n) := \frac{\psi_b(n)}{n \log \log n}$. For $n \to \infty$, we have $\lim R_b(N_n) = \frac{e^\gamma}{\zeta(b)}$.
- **Proposition 2:** Pick a real number $b > 1$. For $n \geq 2$ we have $n^2 > \varphi(n)\psi_b(n) \geq \frac{n^2}{\zeta(b)}$.
- **Theorem 1:** Under RH the ratio $R_b(N_n)$ is $> \frac{e^\gamma}{\zeta(b)}$ for $n \geq 3$. If RH is false, this is still true for infinitely many n.
- **Conjecture 1:** Fix a real number $b > 1$. For all $n \geq 3$ we have $R_b(N_n) > \frac{e^\gamma}{\zeta(b)}$.
- **Theorem 2:** Conjecture 1 is equivalent to RH.

  Hint of the proof: If RH is true, use the first statement of theorem 1. If RH is false, use the function $g(x) := \frac{e^\gamma}{\zeta(b)} \log \theta(x) \frac{\prod_{p \leq x}(1-1/p)}{\prod_{p \leq x}(1-1/p^b)}$.

Summary

Type	Physics	function $\psi_u(n)$	ζ arg. u
Dedekind	mcs of qdits	$\psi(n) = n\prod_{p\mid n} 1+\frac{1}{p}$	2
Gen. Dedek.	phase states in BC	$\psi_b(n) = n\prod_{p\mid n} \frac{1-1/p^b}{1-1/p}$	$b>1$
Goldbach	Goldbach pairs	$\psi_G(2n) = 2C_2 n\prod_{p\mid 2n} \frac{p-1}{p-2}$	$2n\to\infty$

- Comparison of three Dedekind type functions for formulating the equivalence of RH to the inequality $\frac{\psi_u(n)}{n\log\log n} > \frac{e^\gamma}{\zeta(u)}$.