Multi-qudits in Veldkamp space

Michel Planat

Institut FEMTO-ST, CNRS, 15 B Avenue des Montboucons, F-25044 Besançon
(michel.planat@femto-st.fr)

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- A partial linear space is an incidence structure $\Gamma(P, L)$ of points P and lines L satisfying the axioms **(i)** any line is at least with two points **(ii)** any pair of distinct points is incident with at most one line.
- A geometric hyperplane of a partial linear space is a proper subspace meeting each line (necessarily in a unique point or the whole line).
- **Hyperplanes** of a (connected) partial linear space satisfy (and are usefully built from) the following set theoretical equation between hyperplanes H, H' and H''

$$H'' = H \oplus H' = \overline{H \Delta H'}, \text{ for any } H \text{ and } H',$$

where Δ means the **symmetric difference** $(H \cup H') \setminus (H \cap H')$ and the overline symbol means the complement.

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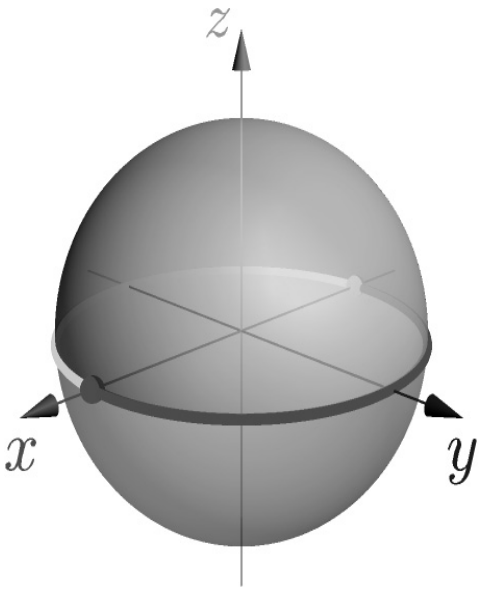
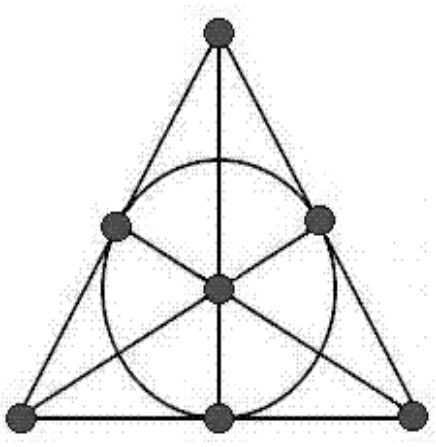
- In the Veldkamp space $\nu(\Gamma)$ of a point-line incidence geometry $\Gamma(P, L)$ of points P and lines L
  (i) a **point** is a geometric hyperplane and
  (ii) a **line** is the collection $H'H''$ of all geometric hyperplanes H of Γ such that $H' \cap H'' = H' \cap H = H'' \cap H$ or $H = H', H''$, where H' and H'' are distinct points of $\nu(\Gamma)$.

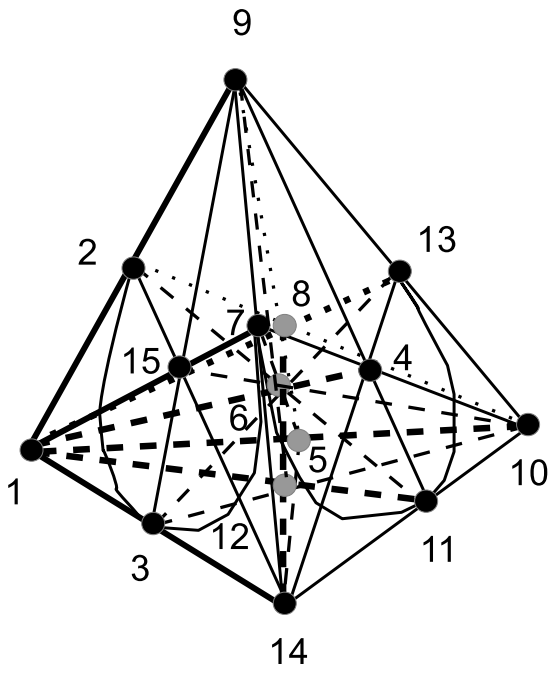


The real projective plane

- A projective space can be thought of as the set of lines through the origin of a vector space V. The cases when $V = \mathbb{R}_2$ and $V = \mathbb{R}_3$ are the real projective line $P^1(\mathbb{R})$ and the real projective plane $P^2(\mathbb{R})$.
- $P^2(\mathbb{R})$ can be described to be the points on the sphere S_2, where every point P and its antipodal point are not distinguished.



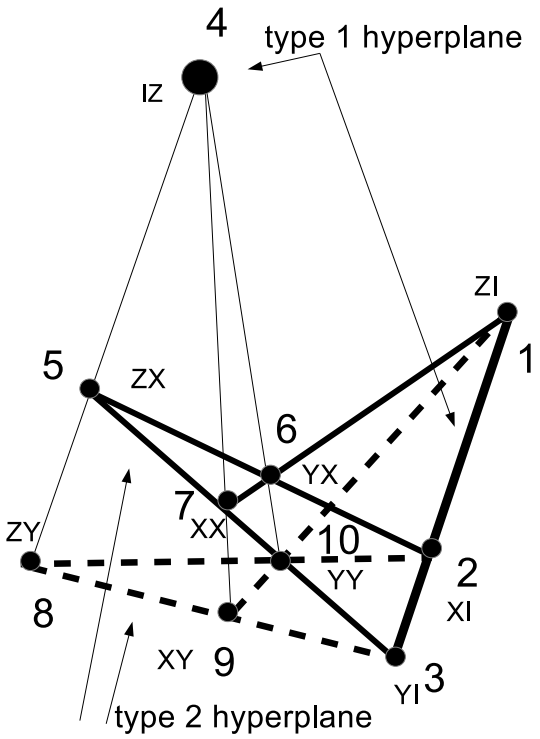
PG(2,2)



PG(3,2)

- A **finite projective space** is a projective space where P is a finite set of points, each line contains the same number of points and the order of the space is defined as one less than this common number. For dimension at least three, Wedderburn's theorem implies that the division ring over which the projective space is defined must be a **finite field**, $GF(q)$, whose order (that is, number of elements) is q (a prime power).

- In the Veldkamp space $\nu(\Gamma)$ of a point-line incidence geometry $\Gamma(P, L)$ of points P and lines L
  (i) a **point** is a geometric hyperplane and
  (ii) a **line** is the collection $H'H''$ of all geometric hyperplanes H of Γ such that $H' \cap H'' = H' \cap H = H'' \cap H$ or $H = H', H''$, where H' and H'' are distinct points of $\nu(\Gamma)$.
- **Proposition**: **The Veldkamp space of a geometry with three points on every line is the finite projective space** $PG(n, 2)$
  (of order $2^n \cdots + 2 + 1 = 2^{n+1} - 1$),
  for some n.

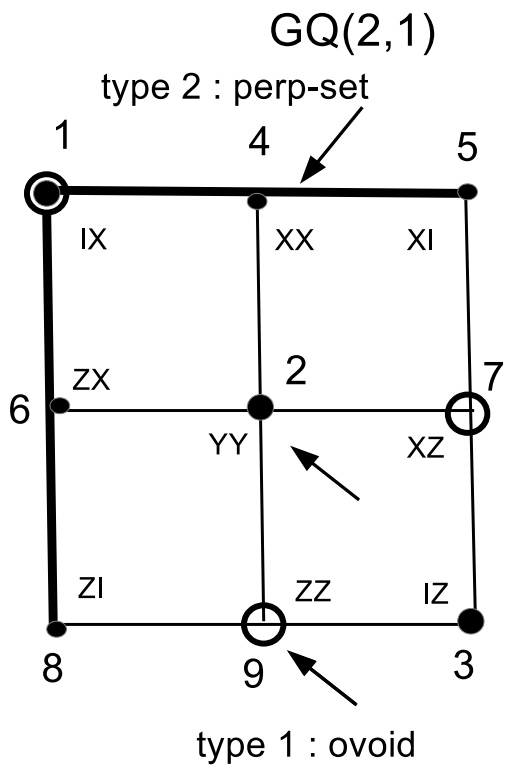


- The Desargues configuration 10_3. There are two types of hyperplanes of type 1 (1 point and one line: 10 copies) and type 2 (four lines: 5 copies). The Veldkamp space is isomorphic to $PG(3,2)$.

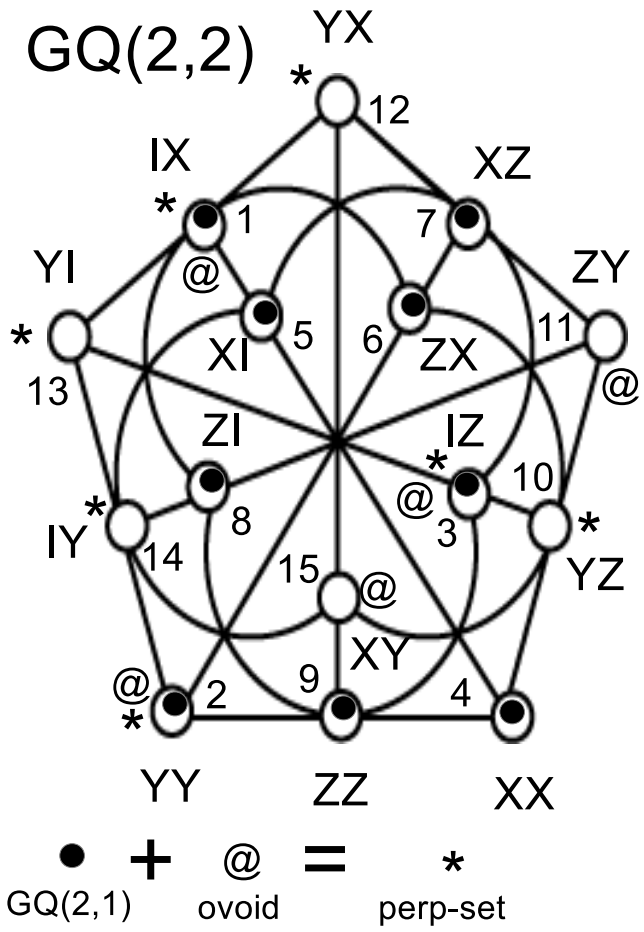
- A **near polygon** is a connected partial linear space S, with the property that given a point x and a line L, there always exists a unique point on L nearest to x.
- A generalized polygon (or generalized n-gon) is an incidence structure between a discrete set of points and lines whose incidence graph has diameter n and girth $2n$ [1].
  The definition implies that a generalized n-gon cannot contain i-gons for $2 \leq i < n$ but can contain ordinary n-gons.
  A generalized polygon of order (s, t) is such that every line contains $s + 1$ points and every point lies on $t + 1$ lines.
  A projective plane of order n is a generalized 3-gon. The generalized 4-gons are the **generalized quadrangles**. Generalized 6-gons, 8-gons, etc are **hexagons, octagons,** etc.
  According to Feit-Higman theorem, finite generalized n-gons with $s > 1$ and $t > 1$ may exist only for $n \in \{2, 3, 4, 6, 8\}$ [2].

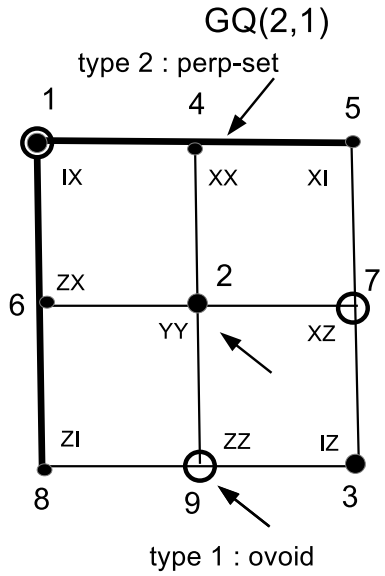
[1] The **diameter** of a graph is the distance between its furthest points. The **girth** is the shortest path from a vertex to itself.

[2] J. Tits and R. M. Weiss, Moufang polygons, Springer Monographs in Mathematics (Springer, Berlin, 2002).



- The generalized quadrangle $GQ(2,1)$. There are two types of hyperplanes of type 'ovoid' (3 non-collinear points): 9 copies) and type 'perp-set' (two lines: 15 copies). The Veldkamp space is isomorphic to $PG(3,2)$.



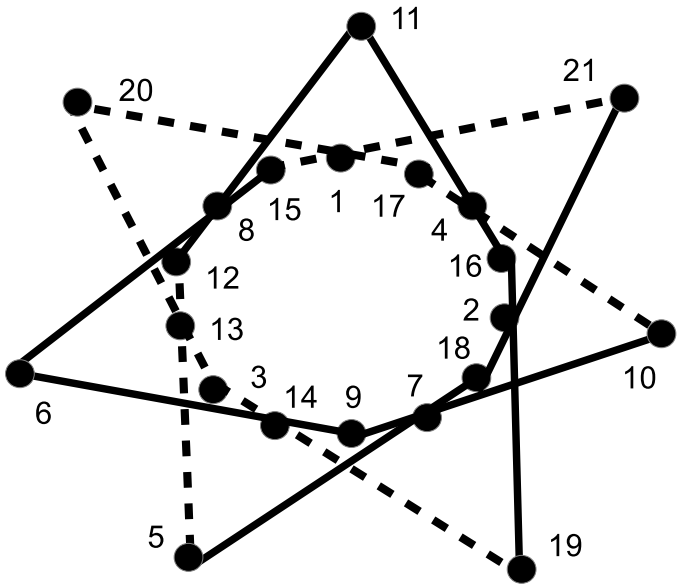


- The generalized quadrangle $GQ(2,2)$. There are three types of hyperplanes of type 'ovoid' (5 non-collinear points: 6 copies), type 'perp-set' (three lines: 10 copies) and type $GQ(2,1)$ (6 lines: 10 copies). Veldkamp space is $PG(4,2)$.

GH(2,1)

21 points, 14 lines

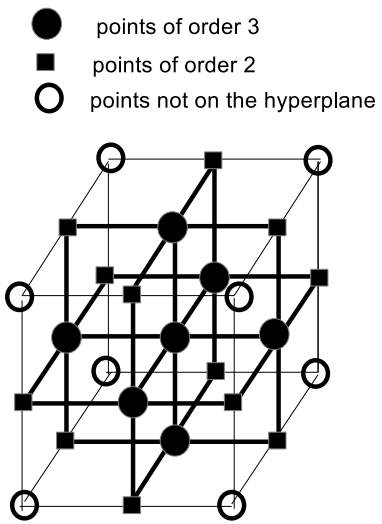
- - - - lines of the singular hyperplane from the point 20



- The generalized (thin) hexagon $GH(2,1)$ and one of its singular hyperplanes.

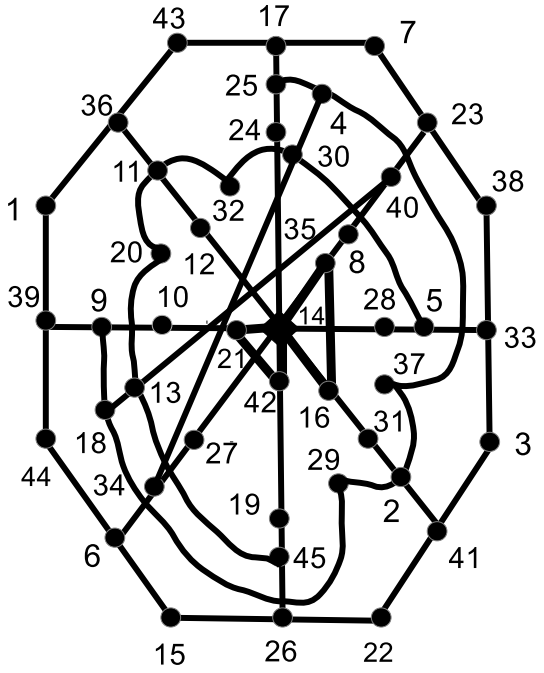
Class	Pts	Lns	n_0	n_1	n_2	Cps	Remark
H1	7	0	7	0	0	24	ovoid
H2a	9	2	3	6	0	56	
H2b	9	2	4	4	1	42	
H3	11	4	2	6	3	84	
H4	13	6	0	8	5	21	singular hyperplane
H5	15	8	0	6	9	28	

- Take the collinearity graph of the $GH(2,1)$ geometry. It has 42 edges and an automorphism group of structure $\mathbb{Z}_2.PSL(2,7)$, of order 336. The **singular type of hyperplane** contains all vertices of the collinearity graph at distance ≤ 2 from a selected vertex. Taking all sums $H_4 \oplus H_4$ over the distinct hyperplanes, one gets all hyperplanes of type H_{2b} (**42** copies) and H_3 (**84** copies). Then, all sums $H_{2b} \oplus H_4$ lead to the **24** hyperplanes of type H_1 and the **56** ones of type H_{2a}; further sums of type $H_1 \oplus H_2$ create the collection of **28** hyperplanes of type H_5.
- The Veldkamp space of $GH(2,1)$ is isomorphic to $PG(7,2)$.



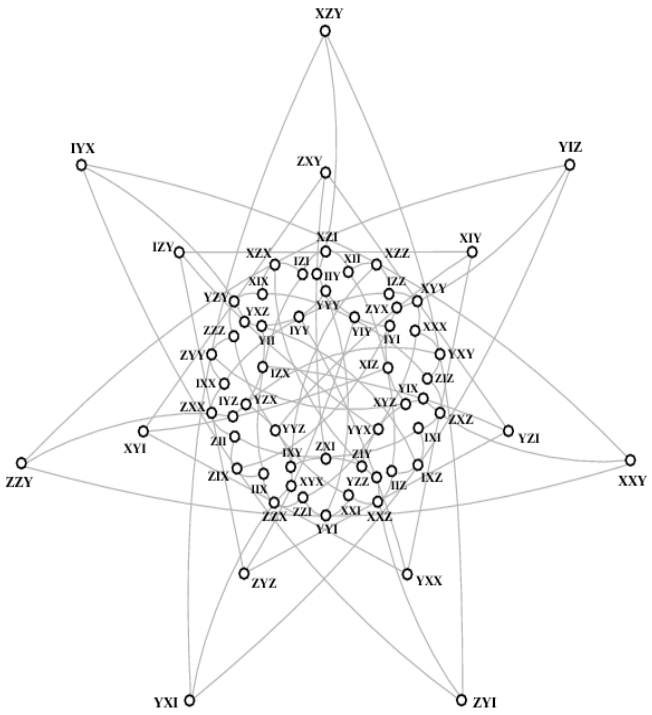
- Type 1 (singular) hyperplane of the $(3 \times 3 \times 3)$ grid. The Veldkamp space is isomorphic to $PG(7, 2)$.

Class	Pts	Lns	n_0	n_1	n_2	n_3	Cps
1	19	15	0	0	12	7	27
2	15	9	0	6	6	3	54
3	13	6	1	6	6	0	108
4	11	3	4	6	0	1	54
5	9	0	9	0	0	0	12



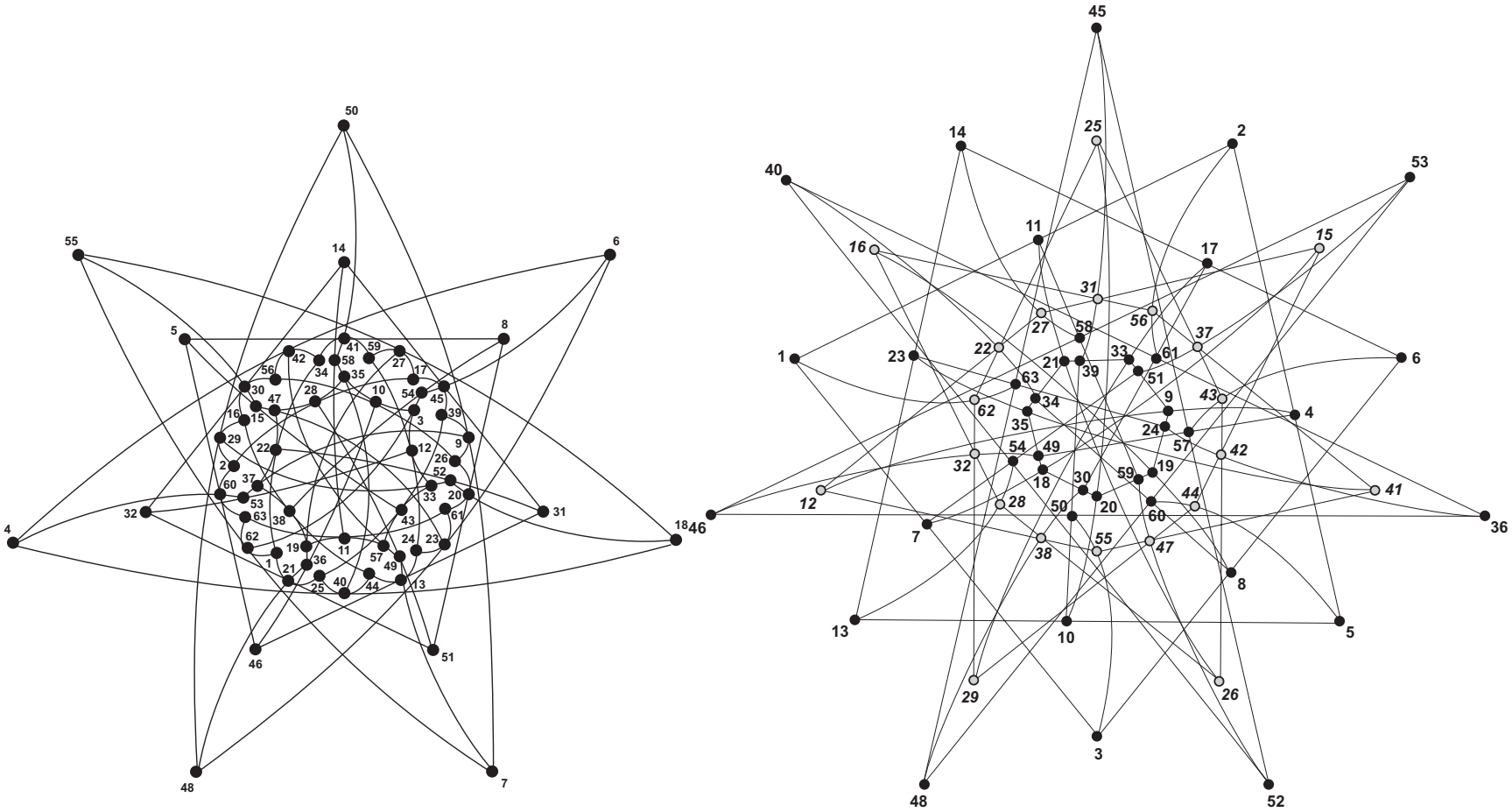
- The generalized octagon $GO(2,1)$. The vertices of the two big triangles at the center of the picture form lines $(8,14,16$ and $(14,21,42)$. They share the point 14 (the big bullet) and the latter does not lie on either straight line around it. For example, one has the lines $(6,34,27)$, $(27,35,8)$ and $(8,40,23)$.

Class	Pts	Lns	Cps	FJ Type
I	15	0	288	$\mathcal{X}_1(15;15,0,0)$
II	17	2	2880	$\mathcal{X}_2(17;11,6,0)$
	17	2	1080	$\mathcal{X}_3(17;12,4,1)$
III	19	4	1440	$\mathcal{X}_4(19;7,12,0)$
	19	4	4320	$\mathcal{X}_5(19;9,8,2)$
	19	4	4320	$\mathcal{X}_6(19;8,10,1)$
	19	4	1440	$\mathcal{X}_7(19;10,6,3)$
IV	21	6	2160	$\mathcal{X}_8(21;5,14,2)$
	21	6	360	$\mathcal{X}_9(21;4,16,1)$
	21	6	2340	$\mathcal{X}_{10}(21;8,8,5)$
	21	6	4320	$\mathcal{X}_{11}(21:7,10,4)$
	21	6	6360	$\mathcal{X}_{12}(21;6,12,3)$
V	23	8	3960	$\mathcal{X}_{13}(23;6,10,7)$
	23	8	5040	$\mathcal{X}_{14}(23;5,12,6)$
	23	8	2880	$\mathcal{X}_{15}(23;3,16,4)$
	23	8	2880	$\mathcal{X}_{16}(23:4,14,5)$
VI	25	10	36	$\mathcal{X}_{17}(25;0,20,5)$
	25	10	5040	$\mathcal{X}_{18}(25;4,12,9)$
	25	10	720	$\mathcal{X}_{19}(25;1,18,6)$
	25	10	1584	$\mathcal{X}_{20}(25;5,10,10)$
	25	10	1440	$\mathcal{X}_{21}(25;3,14,8)$
	25	10	1440	$\mathcal{X}_{22}(25;2,16,7)$
VII	27	12	360	$\mathcal{X}_{23}(27;4,10,13)$
	27	12	2880	$\mathcal{X}_{24}(27;2,14,11)$
	27	12	1440	$\mathcal{X}_{25}(27;3,12,12)$
	27	12	240	$\mathcal{X}_{26}(27;0,18,9)$
VIII	29	14	720	$\mathcal{X}_{27}(29;1,14,14)$
	29	14	225	$\mathcal{X}_{28}(29;0,16,13)$
	29	14	1800	$\mathcal{X}_{29}(29;2,12,15)$
IX	31	16	720	$\mathcal{X}_{30}(31;1,12,18)$
	31	16	360	$\mathcal{X}_{31}(31;2,10,19)$
X	33	18	300	$\mathcal{X}_{32}(33;0,12,21)$



Three-qubit operators in the split-Cayley hexagon GH(2,2)

[3] Three-Qubit Operators, the Split Cayley Hexagon of Order Two and Black Holes, Phys.Rev. D78: 124022, 2008.



- The generalized hexagon $GH(2,2)$ (left) and its dual (right). For both hexagons the Veldkamp space is $PG(13,2)$.

Class	Pts	Lns	DPts	Cps	StGr	FJ Type
I	21	0	0	36	$PGL(2,7)$	$\mathcal{V}_2$(21;21,0,0,0)
II	23	3	1	126	$(4 \times 4) : S_3$	$\mathcal{V}_7$(23;16,6,0,1)
III	25	6	0	504	S_4	$\mathcal{V}_{11}$(25;10,12,3,0)
IV	27	9	0	28	$X_{27}^+ : QD_{16}$	$\mathcal{V}_1$(27;0,27,0,0)
	27	8+1	0	756	D_{16}	$\mathcal{V}_{13}$(27;8,11,8,0)
	27	9	3+1	252	$2 \times S_4$	$\mathcal{V}_8$(27;8,15,0,4)
	27	6+3	0	1008	D_{12}	$\mathcal{V}_{17}$(27;6,15,6,0)
V	29	12	0	1008	D_{12}	$\mathcal{V}_{18}$(29;5,12,12,0)
	29	12	2c	1512	D_8	$\mathcal{V}_{23}$(29;4,16,7,2)
	29	12	2nc	1008	D_{12}	$\mathcal{V}_{19}$(29;6,12,9,2)
	29	12	4	504	S_4	$\mathcal{V}_{12}$(29;7,12,6,4)
VI	31	15	6+1	63	$(4 \times 4) : D_{12}$	$\mathcal{V}_6$(31;0,24,0,7)
	31	15	2+1	1512	D_8	$\mathcal{V}_{24}$(31;4,12,12,3)
	31	15	3	2016	S_3	$\mathcal{V}_{25}$(31;4,12,12,3)
VII	33	18	3+1	1008	D_{12}	$\mathcal{V}_{20}$(33;2,12,15,4)
	33	18	2+2	756	D_{16}	$\mathcal{V}_{14}$(33;4,8,17,4)
VIII	35	21	14	36	$PGL(2,7)$	$\mathcal{V}_3$(35;0,21,0,14)
	35	21	4+2	756	D_{16}	$\mathcal{V}_{16}$(35;0,13,16,6)
	35	21	6	1008	D_{12}	$\mathcal{V}_{21}$(35;2,9,18,6)
IX	37	24	8	756	D_{16}	$\mathcal{V}_{15}$(37;1,8,20,8)
	37	24	6+3+1	1008	D_{12}	$\mathcal{V}_{22}$(37;0,12,15,10)
X	39	27	8+4+1	378	$8 : 2 : 2$	$\mathcal{V}_{10}$(39;0,10,16,13)
XI	43	33	12+3+1	252	$2 \times S_4$	$\mathcal{V}_9$(43;0,3,24,16)
XII	45	36	18	56	$X_{27}^+ : D_8$	$\mathcal{V}_5$(45;0,0,27,18)
XIII	49	42	28	36	$PGL(2,7)$	$\mathcal{V}_4$(49;0,0,21,28)

- Classes of geometric hyperplanes of the split Cayley hexagon $GH(2,2)$.

Class	Pts	Lns	DPts	Cps	StGr	FJ Type
I	23	3	1	252	$Q_8 : S_3$	$\mathcal{W}_3(23;16,6,0,1)$
II	27	9	1	2016	S_3	$\mathcal{W}_{10}(27;8,12,6,1)$
	27	9	0	2016	S_3	$\mathcal{W}_{12}(27;6,15,6,0)$
III	31	15	1+6	63	$X_{32}^{+} : S_3$	$\mathcal{W}_1(31;0,24,0,7)$
	31	15	3(1+2)	378	$(2 \times 4) \cdot 2^2$	$\mathcal{W}_6(31;0,20,8,3)$
	31	15	1+2+3	1008	D_{12}	$\mathcal{W}_8(31;4,15,6,6)$
	31	15	2	3024	2^2	$\mathcal{W}_{13}(31;4,11,14,2)$
	31	15	1+2	3024	2^2	$\mathcal{W}_{14}(31;4,12,12,3)$
IV	35	21	7	864	D_{14}	$\mathcal{W}_7(35;0,14,14,7)$
	35	21	1+2+3	2016	S_3	$\mathcal{W}_{11}(35;2,9,18,6)$
V	39	27	3+6	252	$2 \times S_4$	$\mathcal{W}_4(39;0,6,24,9)$
	39	27	3+9	336	$S_3 \times S_3$	$\mathcal{W}_5(39;0,9,18,12)$
	39	27	3+3+6	1008	D_{12}	$\mathcal{W}_9(39;0,9,18,12)$
VI	47	39	1+6+16	126	$(4 \circ Q_8) : S_3$	$\mathcal{W}_2(47;0,0,24,23)$

- Classes of geometric hyperplanes of the dual of split Cayley hexagon $GH(2,2)$.

Geometry	Pts	Lns	bits	Cls	Remark
5-cell	5	10	**1**	1	
octahedron	6	8	**2**	1	
Fano plane	7	7	**3**	1	smallest projective plane $PG(2,2)$
Pappus	9	9	2	1	
GQ(2,1)	9	6	4	2	alias the (3×3)- grid
$K(3,3,3)$	9	27	6	3	
Desargues	10	10	**4**	2	
GQ(2,2)	15	15	**5**	3	alias Cremona-Richmund
GH(2,1)	21	14	**8**	6	thin generalized hexagon
GQ(2,4)	27	45	6	2	
Pappus 3-fold cov.	27	27	8	9	
$S_{1,1,1}$	27	27	8	5	alias the $(3 \times 3 \times 3)$-grid
$L_3 \times GQ(2,2)$	45	45	**10**	8	near hexagon
$\tilde{GQ}(2,2)$	45	45	**11**	15	
GO(2,1)	45	30	**16**	34	thin generalized octagon
GH(2,2)	63	63	14	25	a 'split Cayley hexagon'
GH(2,2) dual	63	63	14	14	(Frohard, 1994)
$S_{1,1,1,1}$	81	108	16	29	alias the $(3 \times 3 \times 3 \times 3)$-grid
NO(2,4)	315	525	**28**	470	Cohen-Tits near oct. (Bisnoi, 2015)
NH(2,14;2)	759	3795	23	26	M_{24} near hexagon (Brouwer,2000)
GH(2,8)	819	2457	28	?	$3D_4(2)$ hexagon (Frohard, 1992)

general form	Points	Copies	Remark
C_x where $x \in V_n \setminus \{0\}$	$(1/2)4^n - 1$	$4^n - 1$	perp-set
H_x where $Q_0(x) = 0$	$(1/2)(4^n + 2^n) - 1$	$(1/2)(4^n + 2^n)$	
H_x where $Q_0(x) = 1$	$(1/2)(4^n - 2^n) - 1$	$(1/2)(4^n - 2^n)$	

- The three types of hyperplanes for (real) n-qubits. $Q_0(x)$ is the bilinear form defined over the symplectic polar space. The Veldkamp space is $PG(2n, 2)$.

general form	Points	Copies	Remark
C_x where $x \in V_n \setminus \{0\}$	31	63	perp-set
H_x where $Q_0(x) = 0$	35	36	
H_x where $Q_0(x) = 1$	27	28	$GQ(2, 4)$

- The three types of hyperplanes for (real) 3-qubits.

[4] The Veldkamp space of multiple qubits, P. Vrana and P. Lévay, J. Phys. A: Math. Theor. **43** 125303 (2010).

- **unitary groups**
  $U_3(3)$ $U_3(4)$ $U_3(5)$ $U_3(7)$ $\cdots$
  $U_4(2) = Sp(4,3)$ $U_4(3)$ $U_4(4)$ $U_4(5)$
  $U_5(2)$ $\cdots$
  $U_6(2) = Fi_{21} \cdots$
- **geometries**
  $GH(2,2)$ $[208_6, 416_3]$[5] $Hoffmann - Singleton$ $[2107_{21}, 14749_3] \cdots$
  [two QT, $GQ(3,3)$] $GQ(3,3^2), NH(2,14;(2,4))$[6] $GQ(4,4^2)$ $GQ(5,5^2)$
  $GQ(4,8)$ $\cdots$
  $NH(2,20;4)$[7] $\cdots$

[5] The $U_3(4)$ configuration consists of the 208 non-isotropic points of the projective space $\mathbb{P}(U)$ (where U is the 3-dim unitary space over $\mathbb{F}_{16}$) and the corresponding 416 orthogonal bases [Brouwer et al, J. Comb. Theor. A 116, 1056 (2009)].

[6] The $U_4(3)$ **near hexagon** has 567 points, 2835 lines/triangles and the Veldkamp space PG(20,2).

[7] This the **biggest near hexagon** (lines of length 3) with 891 points, 6237 lines/triangles and of Veldkamp space $PG(21,2)$ (Brouwer et al, 1994).

class	Pts	Lns	Cps	Type
I	80	32	195	[80; 0, 64, 16, 0, 0, 0, 0]
II	88	56	208	[88; 0, 12, 75, 0, 0, 0, 1]
III_1	96	80	16640	[96; 3, 15, 30, 30, 15, 3, 0]
III_2	96	80	2496	[96; 10, 5, 25, 50, 0, 1, 5]
IV_1	104	104	6240	[104; 5, 4, 20, 40, 31, 0, 4]
IV_2	104	104	24000	[104; 0, 13, 26, 26, 26, 13, 0]
V_1	112	128	8320	[112; 0, 3, 15, 46, 30, 15, 3]
V_2	112	128	3900	[112; 0, 0, 28, 32, 32, 16, 4]
VI	120	152	3120	[120; 0, 0, 11, 40, 40, 20, 9]
VII	136	200	416	[136; 1, 0, 0, 0, 75, 60]

- The 10 types of hyperplanes of the $U_3(4)$ configuration: the Veldkamp space is $PG(15, 2)$ as for $GH(2, 2)$. All hyperplanes follow from Veldkamp sums starting with the **type II hyperplane**: the latter is defined from points of the graph that are **either at minimum or maximal distance**.