Grothendieck's dessins d'enfants: miscellaneous applications

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A few references on graphs, finite geometries and finite groups

- Andries E. Brouwer, Willem H. Haemers, **Spectra of graphs** http://www.win.tue.nl/ aeb/graphs/srg/srgtab.html (srg graphs up to 1300 vertices)
- Geric Moorhouse, **Generalized polygons** on small order http://www.uwyo.edu/moorhouse/pub/genpoly/
- R. Wilson et al, **ATLAS of Finite Group Representations** - Version 3 http://brauer.maths.qmul.ac.uk/Atlas/v3/ It currently contains information (including 5215 representations) on about 716 groups.

Symplectic groups in Atlas

ATLAS of Finite Group Representations - Version 3

Finding information on a group

Sporadic groups
Alternating groups
Linear groups
Classical groups
Exceptional groups of Lie type
Miscellaneous groups

Symplectic groups
S4(2)' S4(3) *S4(4)* S4(5) S4(7) S4(9)
S4(11) S4(13) S4(17) S4(19)
S6(2) S6(3) S6(5)
S8(2) S8(3)
S10(2)
Unitary groups
U3(3) U3(4) U3(5) U3(7) U3(8) U3(9)
U3(11) U3(13) U3(16)
U4(2) U4(3) U4(4) U4(5)
U5(2) U5(3) U5(4)
U6(2) U6(3)
U7(2)
U8(2)
Orthogonal groups
O7(3)
O8+(2) O8−(2) O8+(3) O8−(3)
O9(3)
O10+(2) O10−(2)

The two-generator free group, its subgroups and the cosets

- Let $F = \langle a, b \rangle$ be the free group on two generators. Elements in the group are **words** u, and $uu^{-1} = e$. We restrict to the free group $G = \langle a, b \mid b^2 = 1 \rangle$. The **index** $n := |G : H|$ of a subgroup H in G counts the number of cosets/copies of H that fill up G. A right coset with respect to an element $g \in G$ is defined as $Hg = \{hg : h \in H\}$. The set of right cosets **partitions** G.
- A transversal is an indexed set of (right) coset representatives for H in G, and the **coset table** expresses the action of a, b on them. (Coxeter-Todd algorithm). Under a and b, the indexed coset representatives are mapped to a two-generator permutation group $P = \langle g_0, g_1 \rangle$.
- The latter is a **map** on a **compact orientable surface** that is, a triple (g_0, g_1, g_∞) with $g_0 g_1 g_\infty = 1$, from which V vertices, E edges and F faces are defined by the cycles of g_0, g_1 and g_∞. The map may be seen as a bicolored **(hyper)map** $\mathcal{D}$ with B black and W white vertices, adjacent vertices always have opposite color and the segments are the n edges. The dessin $\mathcal{D}$ is endowed a natural topological structure with **Euler characteristic** $2 - 2g = B + W + F - n$.

A stabilized point/line (possibly contextual) geometry

- Coset graphs and the related dessins provide a coordinatization to many point-line geometries occuring in the investigation of commutation of quantum observables.

  Taking the **permutation group** P (thus a dessin $\mathcal{D}$) corresponding to a subgroup H of G, one proceeds by first listing the m non-isomorphic subgroups S_m **stabilizing a pair of elements/cosets**.

  Then, given a S_m, all points on a line of the putative geometry $\mathcal{G}_m$ should share the **same** two-point stabilizer subgroup of P. The lines of a $\mathcal{G}_m$ are thus distinguished by their (isomorphic) stabilizers acting on different G-sets. Doing this, the **cosets happen to coordinatize** the edges of the $\mathcal{D}$ and, at the same time, the vertices of the resulting geometries $\mathcal{G}_m$.

- While the commutator for observables A and B is that for a ring: $[A, B] = AB - BA$, the commutator for the representative of two cosets a and b is the group theoretical one: $(a, b) = a^{-1}b^{-1}ab$. We identify a contextual geometry as one where at least one line of l points/cosets of the geometry fails to satisfy $(a_1, a_1, \ldots, a_l) = e$.

Multiple qudits as symplectic polar spaces [1]

- The multiple qudit **Pauli group** $\mathcal{P}_q(q = p^n)$ is the n-fold tensor product between single p-dit Pauli operators with $\omega = \exp(\frac{2i\pi}{p})$, i.e. $IX = I \otimes X$, $IY = I \otimes Y$...

  Observables of $\mathcal{P}_q/\text{Center}(\mathcal{P}_q)$ are seen as the elements of the $2n$-dim. Vector space $V(2n, p)$ defined over $\mathbb{F}_p$. The **commutator** $[.,.] : \ V(2n, p) \times V(2n, p) \to \mathcal{P}'_q$ induces a non-singular alternating bilinear form on $V(2n, p)$, and simultaneously a symplectic form on the projective space $PG(2n-1, p)$ over $\mathbb{F}_p$.

- The $|V(2n, q)| = p^{2n}$ observables of $\mathcal{P}_q/\text{Center}(\mathcal{P}_q)$ are **mapped to the points of the symplectic polar space** $W_{2n-1}(p)$ of cardinality

$$|W_{2n-1}(p)| = \frac{p^{2n}-1}{p-1} \equiv \sigma(p^{2n-1}),$$

  and two elements of $[\mathcal{P}_q/\text{Center}(\mathcal{P}_q), \times]$ commute iff the corresponding points of the polar space $W_{2n-1}(p)$ are collinear.

[1] Saniga M and Planat M 2007 *Adv. Stud. Theor. Phys.* **1** 1-4.

Generators of symplectic polar spaces

- A subspace of $V(2n,p)$ is called **totally isotropic** if the symplectic form vanishes identically on it. The number of such totally isotropic subspaces/generators G (of dimension p^n-1 is

$$|\Sigma(W_{2n-1}(p))| = \prod_{i=1}^{n}(1+p^i).$$

- A **spread** S of a vector space a set of generators partitioning its points. $|S| = p^n+1$ and one has $|V(2n,p)|-1 = |S|\times|G| = (p^n+1)\times(p^n-1) = p^{2n}-1$.

  A **generator** G corresponds to a **maximal commuting set** and a spread S corresponds to a maximum (and complete) set of disjoint maximal commuting sets.

  Two generators in a spread are mutually disjoint and the corresponding maximal commuting sets are **mutually unbiased**.

Three qubit symplectic group $Sp(6,2)$: polar space $W_5(2)$

- $Sp(6,2)$ representation on 63 points[2]:
  $G = \langle x, y | x^2 = y^7 = (xy)^9 = (xy^2)^{12} = (x,(yx)^2y)^2 = (x,y)^3 = (x,y^2)^2 \rangle$
  $H = \text{sub}\langle x, y^2, yxy^{-3}xy \rangle$, $\text{Index}(G,H) = 63$
- dessin of group P of order $1,451,520 = |W'(E_7)|$,
  signature $s = (49,9,7,1)$, cycles $[2^{16}1^{31}, 7^9, 9^7]$,
  collinearity graph of **spectrum** $[30^1, -3^{35}, -5^{27}]$,
  with 135 max. cliques of size 7 corresponding to $W_5(2)$.
- the **configuration with** 315 **lines/triangles is stabilized**,
  it is **maximally contextual**.

[2] Atlas of finite group representations:
http://brauer.maths.qmul.ac.uk/Atlas/v3/

The Veldkamp space of (real) multiqubits [3]

general form	Points	Copies	Remark
C_x where $x \in V_n \setminus \{0\}$	$(1/2)4^n - 1$	$4^n - 1$	perp-set
H_x where $Q_0(x) = 0$	$(1/2)(4^n + 2^n) - 1$	$(1/2)(4^n + 2^n)$	
H_x where $Q_0(x) = 1$	$(1/2)(4^n - 2^n) - 1$	$(1/2)(4^n - 2^n)$	

- The three types of hyperplanes for (real) n-qubits. $Q_0(x)$ is the bilinear form defined over the symplectic polar space. The Veldkamp space is $PG(2n, 2)$.

general form	Points	Copies	Remark
C_x where $x \in V_n \setminus \{0\}$	31	63	perp-set
H_x where $Q_0(x) = 0$	35	36	
H_x where $Q_0(x) = 1$	27	28	$GQ(2, 4)$

- **The three types of hyperplanes for (real) 3-qubits: from the configuration with 63 points and 315 lines/triangles.**

[3] The Veldkamp space of multiple qubits, P. Vrana and P. Lévay, J. Phys. A: Math. Theor. **43** 125303 (2010).

Near (and generalized) polygons

- A **near polygon** is a connected partial linear space S, with the property that given a point x and a line L, there always exists a unique point on L nearest to x.
- A generalized polygon (or generalized n-gon) is an incidence structure between a discrete set of points and lines whose incidence graph has diameter n and girth $2n$ [4].

  The definition implies that a generalized n-gon cannot contain i-gons for $2 \leq i < n$ but can contain ordinary n-gons.

  A generalized polygon of order (s, t) is such that every line contains $s + 1$ points and every point lies on $t + 1$ lines.

  A projective plane of order n is a generalized 3-gon. The generalized 4-gons are the generalized quadrangles. Generalized 6-gons, 8-gons, etc are **hexagons, octagons,** etc.

  According to Feit-Higman theorem, finite generalized n-gons with $s > 1$ and $t > 1$ may exist only for $n \in \{2, 3, 4, 6, 8\}$

[4] The **diameter** of a graph is the distance between its furthest points. The **girth** is the shortest path from a vertex to itself.

Two-qutrit symplectic group $Sp(4,3) = U_4(2)$: $GQ(3,3)\cdots$

- $Sp(4,3)$ representations (Atlas):
  $G = \langle x, y | x^2 = y^5 = (xy)^9 = (x,y)^3 = (x, yxy)^2 \rangle$

Index(G,H)	signature	geometry	max.cliques
27	(15,7,3,2)	**GQ(2,4)**	45 triangles
36	(28,8,4,1)	OA(6,3)	135 K_4
40	(28,3,6,0)	**GQ(3,3)**	40 K_4
40	(24,8,8,1)	GQ(3,3) dual	40 K_4
45	(29,9,7,1)	GQ(4,2) dual	27 of size 5

- Geometries related to $Sp(4,3)$, the bold ones have their lines stabilized by the corresponding dessin.

- **symplectic groups**
  $Sp_4(2)'$ $Sp_4(3)$ $Sp_4(4)$ $Sp_4(5)$[5] $\cdots$
  $Sp_6(2)$ $Sp_6(3)$ $Sp_6(5)$
  $Sp_8(2)$ $Sp_8(3)$ [6]
  $Sp_{10}(2)$
- **multiple qudits**
  [two QB, $GQ(2,2)$] [two QT, $GQ(3,3)$] $GQ(4,4)$ [two 5dit, $GQ(5,5)$] $\cdots$
  [three QB, $W_5(2)$] [three QT, $W_5(3)$] [three 5dit, $W_5(5)$]
  [four QB, $W_7(2)$] [four QT, $W_7(3)$]
  [five QB, $W_9(2)$]

[5] Dessin of index 156, signature $(84, 56, 12, 3)$ and cycles $[2^{72}1^{12}, 3^{50}1^{6}, 13^{12}]$ and for the dual $s = (82, 52, 12, 6)$ and cycles $[2^{74}1^{8}, 3^{52}, 13^{12}]$. The two graphs are isospectral.

[6] Dessin of index 3280, signature $s = (1680, 914, 236, 226)$ and cycles $[2^{1600}1^{80}, 4^{728}2^{182}1^{4}, 14^{234}2^{2}]$.

Geometries from of unitary groups

- **unitary groups**
  $U_3(3)$ $U_3(4)$ $U_3(5)$ $U_3(7)$ $\cdots$
  $U_4(2) = Sp(4,3)$ $U_4(3)$ $U_4(4)$ $U_4(5)$
  $U_5(2)$ $\cdots$
  $U_6(2) = Fi_{21} \cdots$
- **geometries**
  $GH(2,2)$ $[208_6, 416_3]$[7] $Hoffmann - Singleton$ $[2107_{21}, 14749_3] \cdots$
  [two QT, $GQ(3,3)$] $GQ(3,3^2), NH(2,14;(2,4))$[8] $GQ(4,4^2)$ $GQ(5,5^2)$
  $GQ(4,8)$ $\cdots$
  $NH(2,20;4)$[9] $\cdots$

[7] The $U_3(4)$ configuration consists of the 208 non-isotropic points of the projective space $\mathbb{P}(U)$ (where U is the 3-dim unitary space over $\mathbb{F}_{16}$) and the corresponding 416 orthogonal bases [Brouwer et al, J. Comb. Theor. A 116, 1056 (2009)].

[8] The $U_4(3)$ **near hexagon** has 567 points, 2835 lines/triangles and the Veldkamp space PG(20,2).

[9] This the **biggest near hexagon** (lines of length 3) with 891 points, 6237 lines/triangles and of Veldkamp space $PG(21,2)$ (Brouwer et al, 1994).

The Veldkamp space of the $U_3(4)$ configuration $[208_6, 416_3]$

class	Pts	Lns	Cps	Type
I	80	32	195	[80; 0, 64, 16, 0, 0, 0, 0]
II	88	56	208	[88; 0, 12, 75, 0, 0, 0, 1]
III_1	96	80	16640	[96; 3, 15, 30, 30, 15, 3, 0]
III_2	96	80	2496	[96; 10, 5, 25, 50, 0, 1, 5]
IV_1	104	104	6240	[104; 5, 4, 20, 40, 31, 0, 4]
IV_2	104	104	24000	[104; 0, 13, 26, 26, 26, 13, 0]
V_1	112	128	8320	[112; 0, 3, 15, 46, 30, 15, 3]
V_2	112	128	3900	[112; 0, 0, 28, 32, 32, 16, 4]
VI	120	152	3120	[120; 0, 0, 11, 40, 40, 20, 9]
VII	136	200	416	[136; 1, 0, 0, 0, 75, 60]

- The 10 types of hyperplanes of the $U_3(4)$ configuration: the Veldkamp space is $PG(15, 2)$ as for $GH(2, 2)$. All hyperplanes follow from Veldkamp sums starting with the **type II hyperplane**: the latter is defined from points of the graph that are **either at minimum or maximal distance**.

- **orthogonal groups**
  $O_7(3) = Sp(6,3)$
  $O_8^+(2)$ $O_8^-(2)$
  $O_8^+(3)$ $O_8^-(3)$
  $O_{10}^+(2)$ $O_{10}^-(2)$
- **dual graphs of multiple qudits** [10]
  [three QT, $W_5(3)]_{index}$ 364
  [(three QB)*, pg(8,7,4)]$_{index}$ 135 pg(6,8,3)$_{index}$ 119
  [(three QT)*, pg(27,26,18)]$_{index}$ 1120 pg(12,27,4)$_{index}$ 1066
  [(four QB)*, pg(?)]$_{index}$ 2295 pg(14,16,7)$_{index}$ 495

[10] A partial geometry pg(s,t,α) is a partial linear space of order (s,t) such that for each anti-flag (x,L) the incidence number $\alpha(x,L)$, being the number of points on L collinear with x, is a constant $\alpha \neq 0$. (Bose 1963).

- With the coset approach in hands, what basic features should possess a subgroup H of G to generate a **contextual** $\mathcal{G}$? We show that the generic 'contextual subgroups' are Kleinian, non-compact, arithmetic and arise from two elliptic generators. The work of C. Maclachlan and G. J. Martin establishes that there are 21 such Kleinian groups K (herafter denoted the MM-census)[11]. We examine the small index subgroups of K in a few classes of the MM-census and how they help to clarify our topic within the new frame of 3-**orbifolds** $\mathbb{H}^3/K$.

[11] C. Maclachlan, and G. J. Martin, The non-compact arithmetic generalized triangle groups, *Topology* **40** 927-944 (2001).

Kleinian groups of the MM-census 1

- A Kleinian group K is a discrete subgroup of $PSL(2,\mathbb{C})$, the group of all **orientation-preserving isometries** of the 3-dimensional space $\mathbb{H}^3$. A **non-compact group** K is such that the corresponding orbifold $\mathbb{H}^3/K$ is non-compact. A Kleinian group K is called **arithmetic** if it is commensurable with the group of units of an order of quaternion algebra A ramified at all real places over a number field k with exactly one complex place.
- Arithmetic Kleinian groups have finite covolume. There are **finitely many two-generator** Kleinian groups $K = \langle f, g \rangle$ that are **non-compact and arithmetic** provided the isometries f and g are **elliptic**, with degrees p and q respectively, satisfying $\mathrm{tr}^2(f) = 4\cos^2(\pi/p)$, $\mathrm{tr}^2(g) = 4\cos^2(\pi/q)$ with parameter $\gamma = \mathrm{tr}[f,g] - 2$.

Kleinian groups of the MM-census 2

- It is known that a quadratic imaginary number field $kK = \mathbb{Q}(\sqrt{-d})$ and a quaternion algebra $AK \equiv M_2(\mathbb{Q}(\sqrt{-d}))$ is associated to a **non-compact arithmetic Kleinian group** K. It is also known that K is commensurable, up to conjugacy, with the **Bianchi group** $PSL(2, O_d)$, where O_d is the ring of integers in $\mathbb{Q}(\sqrt{-d})$. For groups K of the MM-census, $d \in \{1, 3\}$.
- A few classes of Kleinian groups $K^{(i)}$ in the MM-census.

group	p	q	d	γ	covol.	$\mathcal{G}$
$K^{(4)}$	2	4	1	$-1+i$	0.4579	Hesse, Petersen
$K^{(19)}$	4	6	3	-1	0.2114	$GQ(2,1)$ (Mermin sq.)
$K^{(1)}$	2	3	3	$(-3+\sqrt{3}i)/2)$	0.3383	Mermin pent., $GH(2,1)$
$K^{(5)}$	2	4	1	$-2+i$	0.9159	$GO(2,1)$, "$GO(2,4)$"
$K^{(2)}$	2	3	3	$(-1+\sqrt{3}i)/2$	0.6766	$GH(2,1)$, "J_2"

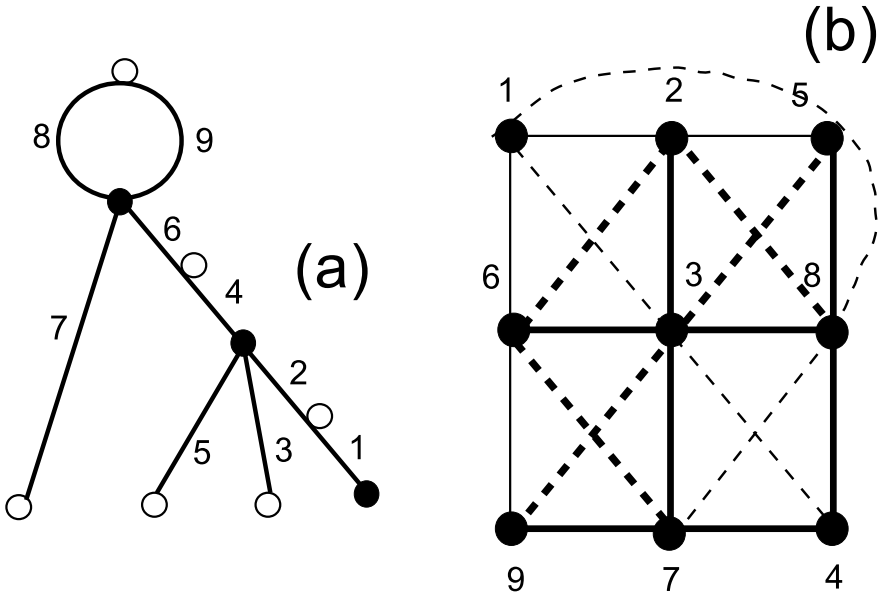
Thin generalized polygons

- Many of the generic **contextual configurations filtered in the MM-census** are 'thin' generalized polygons. A generalized polygon (or generalized n-gon) is an incidence structure between a discrete set of points and lines whose **incidence graph has diameter** n **and girth** $2n$ [12] .
- A **generalized polygon of order** (s, t) is such that every line contains $s+1$ points and every point lies on $t+1$ lines. A projective plane of order n is a generalized 3-gon. The generalized 4-gons are the generalized **quadrangles**. Generalized 5-gons, 6-gons, etc are also called generalized **pentagons, hexagons, octagons**, etc.
- According to Feit-Higman theorem, finite generalized n-gons with $s > 1$ and $t > 1$ may exist only for $n \in \{2, 3, 4, 6, 8\}$ [13]. Below we will meet $GQ(2, 1)$, $GH(2, 1)$ and $GO(2, 1)$.

[12] B. Polster, A. E. Schroth and H. Van Maldeghem, Generalized flatland, *Math. Intelligencer* **23**(4), 33–47 (2001).

[13] J. Tits and R. M. Weiss, *Moufang polygons*, Springer Monographs in Mathematics (Springer, Berlin, 2002).

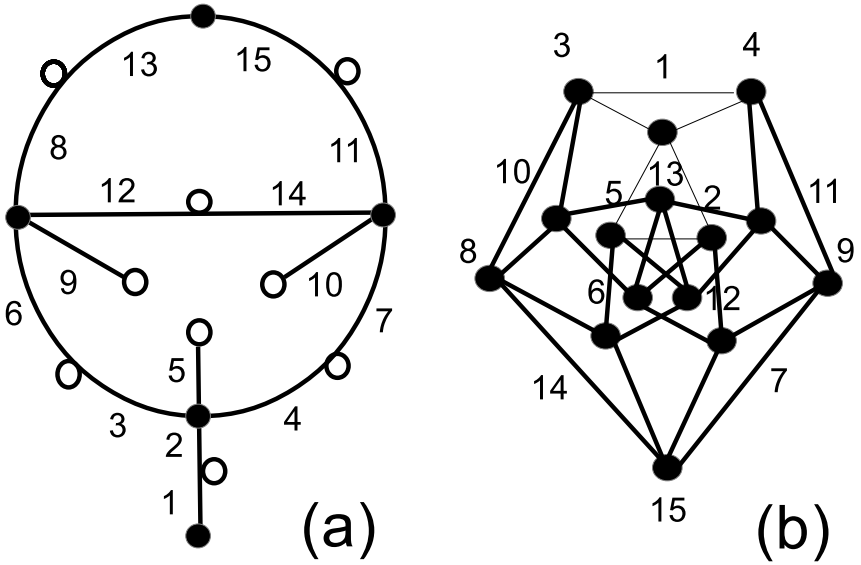
$K^{(4)}$: the Hesse configuration



$$K^{(4)} = \langle x, y | y^2 = x^4 = [(yx^{-1})^2(y^{-1}x)^2]^2 = 1 \rangle .$$

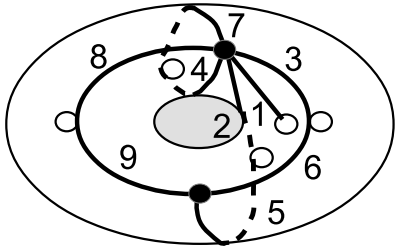
The dessin d'enfant (a) arising from the first subgroup of index 9 of the Kleinan group $K^{(4)}$ and the Hesse configuration (b) that it stabilizes. The coset representatives of the subgroup $K^{(4)}$ in the free group G are: $1 = e,\ 2 = y,\ 3 = yx,\ 4 = yx^{-1},\ 5 = yx^2,\ 6 = yx^{-1}y,\ 7 = yx^{-1}yx,\ 8 = (yx^{-1})^2,\ 9 = yx^{-1}yx^2.$

$K^{(4)}$: the line graph of Petersen graph



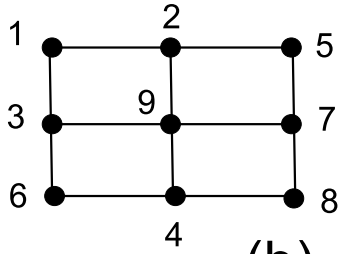
The dessin d'enfant (a) arising from the first subgroup of index 15 of the Kleinan group $K^{(4)}$ and the line graph of the Petersen graph (b) that it stabilizes. The coset representatives are $\{1, 2, \cdots, 15\} = \{e, y, yx, yx^{-1}, yx^2, yxy, yx^{-1}y, (yx)^2, yxyx^{-1}, yx^{-1}yx, (yx^{-1})^2, yxyx^2, yxyx^{-1}y, yx^{-1}yx^2, yx^{-1}yxy\}$.

$K^{(19)}$: Mermin's square $GQ(2,1)$

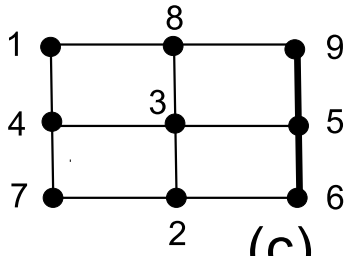


P=<(1,2,4,8,7,3)(5,9,6),
(2,5)(3,6)(4,7)(8,9)>

(a)



(b)

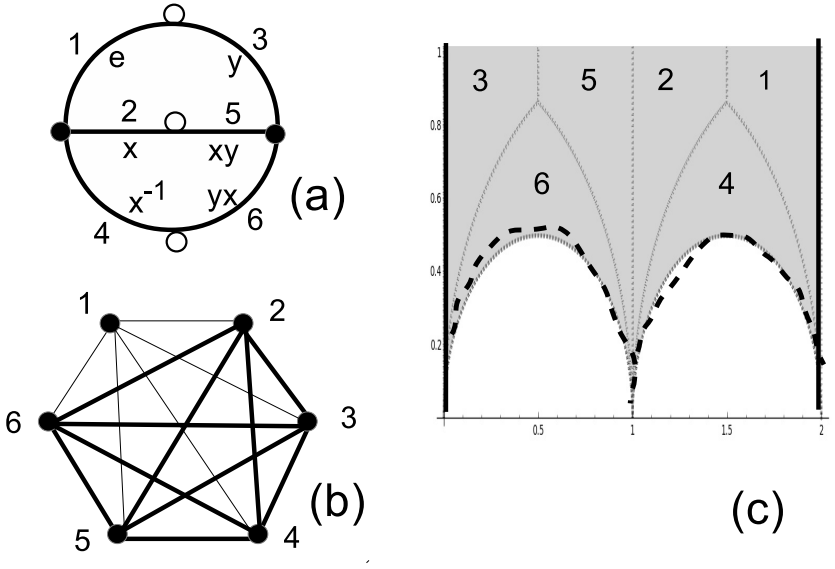


(c)

$$K^{(19)} = \langle x, y | y^4 = x^6 = [x,y]^3 = ([y,x] * y)^2 = (y^{-1} * [y,x])^2 = (x^{-1} * [y,x] * y)^2 = 1 \rangle .$$

The dessin d'enfant (a) arising from the subgroup of index 9 and permutation group $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2^2$ in the Kleinian group $K^{(19)}$, the non-contextual grid (b) and the contextual one (c). The coset representatives are$[1, 2, \cdots, 9] = [e, x, x^{-1}, x^2, xy, x^{-1}y, x^{-2}, x^3, xyx]$

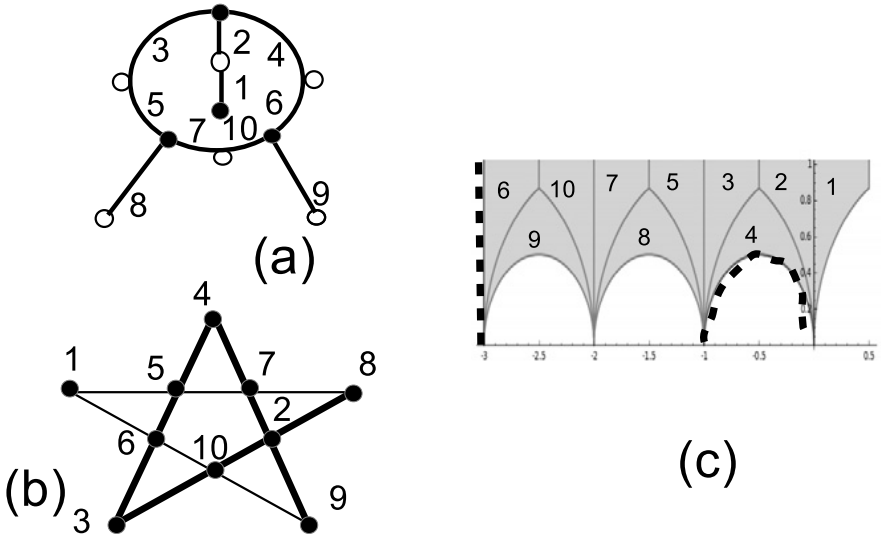
$K^{(1)}$: the congruence subgroup $\Gamma(2)$



$$K^{(1)} = \left\langle x, y | y^2 = x^3 = [(yx^{-1})^2(y^{-1}x)^2]^3 = 1 \right\rangle .$$

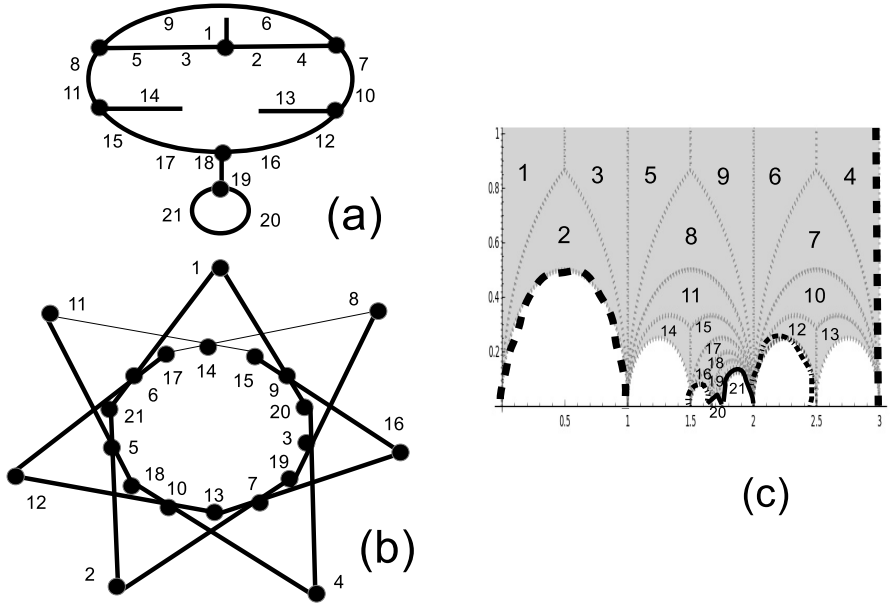
The dessin in (a) from the congruence subgroup of index 6 of type $\Gamma(2)$ of the Kleinian group $K^{(1)}$. This dessin stabilizes the (maximally contextual) complete graph K_6 shown in (b). The corresponding polygon $\Gamma(2)$ is shown in (c). The Belyi function is $f(x) = \frac{4}{27}j(x)$, where $j(x) = \frac{(1-x+x^2)^3}{x^2(x-1)^2}$ is the **modular invariant**.

$K^{(1)}$: Mermin's pentagram



The dessin in (a) arising from the unique subgroup of index 10 isomorphic to S_5 of the Kleinian group $K^{(1)}$. This dessin stabilizes Mermin's pentagram shown in (b). As it is modular the dessin can also be seen as a tiling of the upper-half plane in (c). The coset representatives are $[1, 2, \cdots, 10] = [e, y, yx, yx^{-1}, yxy, yx^{-1}y, (yx)^2, yxyx^{-1}, yx^{-1}yx, (yx^{-1})^2]$.

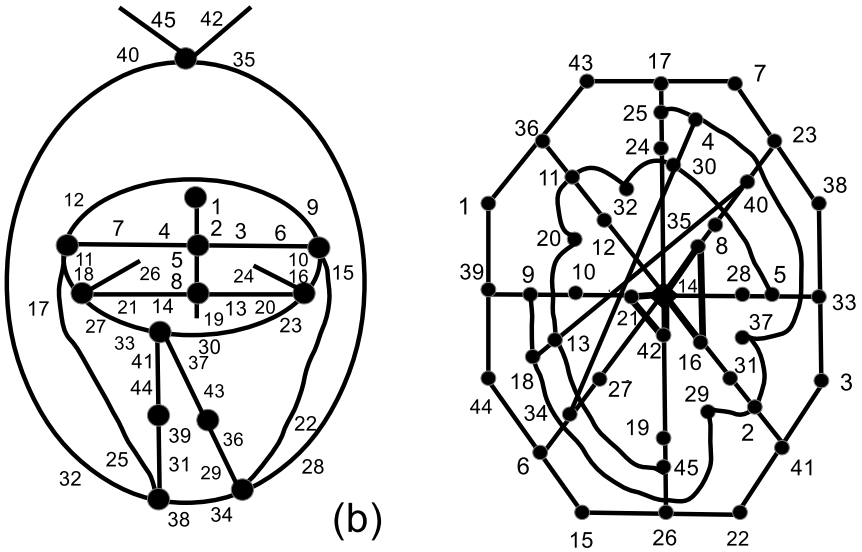
$K^{(1)}$: the thin hexagon $GH(2,1)$



The dessin in (a) arising from the unique subgroup of index 21 isomorphic to $SL(2,7)$ of the Kleinian group $K^{(1)}$. This dessin stabilizes the thin generalized hexagon $GH(2,1)$ shown in (b).

The tiling of the upper-half plane is in (c). The coset representatives are $[1..21]=$ $[e, x, x^{-1}, xy, x^{-1}y, xyx, xyx^{-1}, y^{x}, x^{-1}yx^{-1}, xyx^{-1}y, x^{-1}yxy, xyx^{-1}yx, xyx^{-1}yx^{-1}, x^{-1}(yx)^2, x^{-1}yxyx^{-1}, xyx^{-1}yxy, x^{-1}yxyx^{-1}y, xyx^{-1}(yx)^2, xyx^{-1}(yx)^2y, xyx^{-1}(yx)^3, xyx^{-1}(yx)^2yx^{-1}]$.

$K^{(5)}$: the thin octagon $GO(2,1)$



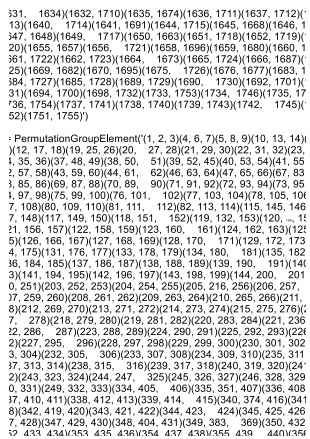
$$K^{(5)} = \langle x, y | y^2 = x^4 = [yxy^{-1}xyx^{-1}]^4 = 1 \rangle .$$

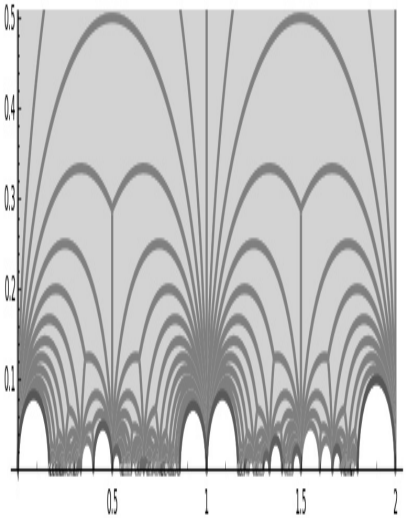
The dessin in (a) arising from the unique subgroup of index 45 isomorphic to A_6 of the Kleinian group $K^{(5)}$. This dessin stabilizes the thin generalized octagon $GO(2,1)$ and the cosets from 1 to 45 are

$[e, y, yx, yx^{-1}, yx^2, yxy, yx^{-1}y, yx^2y, (yx)^2, yxyx^{-1}, yx^{-1}yx, (yx^{-1})^2, yx^2yx, yx^2yx^{-1},$
$yxyx^2, yxyx^{-1}y, yx^{-1}yx^2, yx^{-1}yxy, (yx^2)^2, yx^2yxy, yx^2yx^{-1}y, yxyx^2y, yxyx^{-1}yx,$
$yxyx^{-1}yx^{-1}, yx^{-1}yx^2y, yx^{-1}yxyx, yx^{-1}yxyx^{-1}, yxyx^2yx, yxyx^2yx^{-1}, yxyx^{-1}yxy,$
$yx^{-1}yx^2yx, yx^{-1}yx^2yx^{-1}, yx^{-1}yxyx^{-1}y, yxyx^2yx^2, yxyx^2yxy, yxyx^2yx^{-1}y,$
$yxyx^{-1}yxyx, yx^{-1}yx^2yx^2, yx^{-1}yx^2yxy, yx^{-1}yx^2yx^{-1}y, yx^{-1}yxyx^{-1}yx^{-1}, yxyx^2yxyx^{-1},$
$yxyx^2yx^{-1}yx, yx^{-1}yx^2yxyx, yx^{-1}yx^2yx^{-1}yx].$

The Ree-Tits octagon $GO(2,4)$: slide 1

Ree-Tits hyperbolic polygon
with 1755 tiles





$$G_1 = \langle x, y | x^2 = y^3 = (xy)^{13} = [x,y]^5 = [x, yxy]^4 = [(xy)^4 (xy^{-1}]^6 = 1 \rangle)$$
$$H_1 = \mathrm{sub} \left\langle G_1 | x = y^{-1}(xy)^2 xy^{-1}(xy)^3 (xy^{-1})^2 = 1 \right\rangle .$$

The index of H_1 in G_1 is 1755 and the corresponding permutation group is the Tits group T of order $17,971,200$. The signature of the dessin is $(B, W, F, g) = (1846, 1170, 270, 113)$ with cycles $[2^{1664}1^{82}, 3^{1170}, 13^{270}]$ R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray and R. Abbott *ATLAS of Finite Group Representations - Version 3*, available at http://brauer.maths.qmul.ac.uk/Atlas/v3/exc/TF42/ .

The Ree-Tits octagon $GO(2,4)$: slide 2

- Finally, the stabilizer subgroup of order 2^{10} of T is used to recover **Ree-Tits octagon** $GO(2,4)$ that has 1755 vertices, 2925 lines/triangles and a collinearity graph of spectrum $[10^1, 5^{351}, 1^{650}, -3^{675}, -5^{78}]$. The subgroup H_1 is found to **contain a 'kernel' which is the Kleinian group** $K^{(1)}$ of index $4,492,800 = 2560 \times 1755$ in G_1. Schematically

$$G \supset G_1 \supset_T \text{from index } {}_{1755} \; H_1 \supset_{2560} K^{(1)}.$$

  The thin octagon $GO(2,1)$ is embedded in $GO(2,4)$ in a unique way [14] . But the supset structure above does not reflect that $GO(2,4) \supset GO(2,1)$ since $GO(2,1)$ follows from the Kleinian group $K^{(5)}$ not $K^{(1)}$.

- Similar chains of subgroups

$$G \supset G_2 \supset_{J_2} \text{from index } {}_{100} \; H_2 \supset_{3024} K^{(2)}.$$

$$G \supset G_2 \supset_{J_2.2} \text{from index } {}_{315} \; H_2'.$$

  occur for building the **Cohen-Tits near octagon** [15] on 315 points and 525 triangles from $J_2.2$ (J_2 the Hall Janko group of order $604,800$).

[14] B. De Bruyn, The uniqueness of a certain generalized octagon of order $(2,4)$.
[15] A. M. Cohen and J. Tits, *Europ. J. Combinatorics* **6** 13-27 (1985).

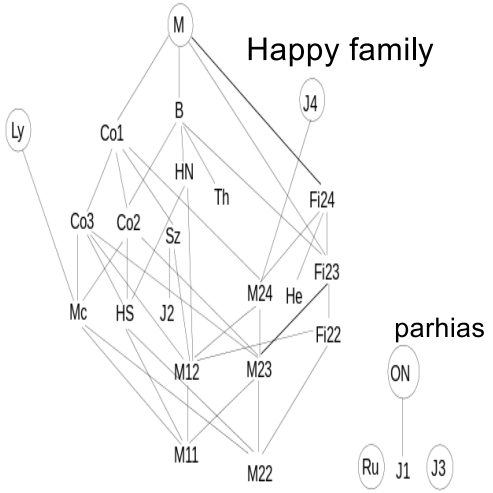
Summary of this section

The **coset parametrization of quantum observables** (from a subgroup H of the free group $G = \langle x, y \rangle$ and the dessin d'enfant methodology) allowed us arrive at the result that **geometric contextuality fits quantum contextuality**. Here we have shown that non-compact arithmetic Kleinian groups $K = \langle f, g \rangle$ generated by two elliptic isometries f and g act as '**contextuality filters**'.
Thin generalized polygons $GQ(2,1)$, $GH(2,1)$ and $GO(2,1)$ (and a few extra cases) when stabilized thanks to the relevant finite index subgroups of Kleinian groups $K^{(i)}$ in the Maclachlan-Martin census have been shown to be **maximally contextual**.
All hyperbolic orbifolds $\mathbb{H}^3/K^{(i)}$ are candidates for a re-examination of geometric contextuality with the further remark that the $K^{(i)}$'s **are commensurable with Bianchi groups** $PSL(2, O_d)$, $d \in \{1, 3\}$ whose elements are in the ring of integers of $\mathbb{Q}(\sqrt{-d})$.

Moonshine [16]

Dyson's words in 1981: *I have a sneaking hope, a hope unsupported by any facts or any evidence, that sometime in the twenty-first century physicists will stumble upon the Monster group, built in some unsuspected way into the structure of the universe. This is of course only a wild speculation, almost certainly wrong.*

1965-1975



Mathieu, 1860

Subquotient relations
between the sporadic groups

The Monster group

- The **Monster group** $\mathbb{M}$ is of cardinality $|\mathbb{M}| = 2^{46}\ 3^{20}\ 5^{9}\ 7^{6}\ 11^{2}\ 13^{3}\ 17\ 19\ 23\ 29\ 31\ 41\ 47\ 59\ 71 \sim 10^{54}$ -this would correspond to the mass in Kg of the known universe.
- Griess (1982) constructed M as the automorphism group of the **Griess algebra, a 196884-dimensional commutative nonassociative algebra**. John Conway (1985) and Jacques Tits (1984, 1985) subsequently simplified this construction.

  The **smallest permutation** representation of the monster is on $2^{4}\ 3^{7}\ 5^{3}\ 7^{4}\ 11\ 13^{2}\ 29\ 41\ 59\ 71$ (about 10^{20} points).
- **Monstrous moonshine**, or moonshine theory, is a term devised by John Conway and Simon P. Norton in 1979, used to describe the unexpected **connection between the monster group M and modular functions**, in particular, **the j function**. It is now known that lying behind monstrous moonshine is a certain conformal field theory having the monster group as symmetries (Borcherds, 1992).

Moonshine and the j-function

- $j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$, with $q = e^{2i\pi\tau}$
- The (unexpected) coincidence

$$196884 = 1 + 196883,$$
$$21493760 = 1 + 196883 + 21296876,$$

  and so on, in which the numbers at the left column occur in **the q-expansion of the modular j-invariant** ($q = e^{2i\pi\tau}$ and τ in the upper-half plane) and the numbers at the right column are **sums of dimensions of the smallest irreducible representations of the Monster** $\mathbb{M}$ [17].
- Moonshine is not limited to the monster, but that similar phenomena may be found for other sporadic groups (generalized moonshine).

[17] T. Gannon, Monstrous moonshine: the fist twenty-five years, *Bull. London Math. Soc.* **38** 1 (2005).